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PREFACE.

THE method of expansions, developed by Lagrange in his *Théorie des Fonctions Analytiques*, has been for many years almost exclusively adopted in this University for the demonstration of the formulæ of the differential calculus. The great name of the originator of this system gave a certain permanency to the method which probably it would not have possessed had it emanated from one less illustrious. Not to insist upon the doubts which have been thrown by several recent writers on the validity of conclusions deduced from the comparison of infinite series, it is certain that the absence of any notion of limits in the algebraical theory of derived functions, gives rise to an entire want of homogeneity between its fundamental conceptions and those which present themselves in its most interesting applications. Within the last few years an endeavour to re-establish the system of limits, has been made by several elementary writers in France, among whom may be mentioned Moigno, Duhamel, and Cournot, and by Professors De Morgan and O'Brien in England. From my own strong conviction of the marked advantage which the method of limits possesses over that of derived functions, both abstractedly and in its applications, and, trusting to the valuable opinion of many writers of the present day on the comparative merits of the two systems, I have been induced to enter upon this treatise. My object has been to present to the English student a work from which he may acquire a thorough and systematic knowledge of the abstract theory of limits, and of its applications to certain branches of coordinate geometry. How far I may have succeeded in this attempt, and of the liability to failure in a work of such a nature I am fully sensible, will be determined by the judgment of the reader.

In the composition of this work, which was commenced in the April of this year, I have derived great assistance from Moigno's *Leçons de Calcul Différentiel et de Calcul Intégral,* Duhamel's *Cours d'Analyse,* and Cournot's *Théorie des Fonctions:* from the last of which treatises I have made considerable extracts in Chapter VII. on the Development of Functions.

Cambridge, November, 1845.

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ERRATA.

Page	line	Error	Correction.
65	7 from bottom	ϕ	r
93	1	a^{xm}	a^{x^m}
—	7 from bottom	$f(x, y) + \Delta F(x, y)$	$f(x, y) + \Delta f(x, y)$
112	13 „	y	r
130	5	h^r	h^ν
141	4	R^r	R_r
144	7 from bottom	$f'(0)$	$f(0)$
148	12	$\frac{1}{1.2.3}\left(h\frac{d}{dx} + k\frac{d}{dy}\right)$	$\frac{1}{1.2.3}\left(h\frac{d}{dx} + k\frac{d}{dy}\right)^3$
—	9 from bottom	ε^{dx+d_y}	$\varepsilon^{d_x+d_y}$
242	3	$-\frac{r^2 d\theta}{dr} = a$	$\frac{r^2 d\theta}{dr} = -a$
—	6	negative	positive
—	7	positive	negative
—	17	$\frac{1}{2} a\theta^3$	$-\frac{1}{2} a\theta^3$
—	18	$v = \frac{1}{2} a$	$v = -\frac{1}{2} a$
—	18	$v = -\frac{1}{2} a$	$v = \frac{1}{2} a$

DIFFERENTIAL CALCULUS.

FIRST PART.

GENERAL PRINCIPLES AND ANALYTICAL APPLICATIONS.

CHAPTER I.

FUNCTIONS.

1. If any two quantities are so connected that any variation in the magnitude of the one implies a corresponding variation in the magnitude of the other, either of these quantities is said to be a *function* of the other. Such a connection is expressed algebraically by means of an equation involving the symbols of the two quantities and any other symbols of invariable magnitudes. Thus x and y are functions the one of the other in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

or

$$y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (2),$$

where a and b are supposed to represent invariable magnitudes.

2. If y be expressed directly in terms of x, as in (2), it is said to be an *explicit* function of x; if it be merely connected

with x by an unsolved equation, as in (1), it is said to be an *implicit* function of x.

3. The connection between x and y may be expressed generally by an equation

$$F(x, y) = 0 \text{ ....(3)},$$

where $F(x, y)$ denotes any expression whatever involving x, y, and constants. These constants are usually called *parameters*, a term borrowed from the theory of conic sections, where the word *parameter* is used to denote a certain fixed line. If we wish to represent in the most general manner that y is an explicit function of x, we may write

$$y = f(x) \text{ ....(4)},$$

$f(x)$ denoting any algebraical expression which involves x and parameters. The equation (4) is easily seen to be a particular form of the equation (3); for we may write it thus,

$$y - f(x) = 0,$$

$y - f(x)$ being merely a particular instance of the general form $F(x, y)$.

4. Functions may be termed *mathematical* or *empirical;* mathematical, if the functionality is established by definition; empirical, if discovered by observation. As an instance of the latter functionality, let y denote the attraction of the Sun upon the Earth, and x the distance between these two masses; then it is known by observation that

$$y = \frac{\mu}{x^2},$$

μ being a constant quantity: y is in this case an empirical function of x.

5. A function $f(x)$ is said to be *continuous* when, as x increases continuously, $f(x)$ passes continuously from one possible value to another through all intervening values: the function is said to be *discontinuous* whenever this condition is violated. Take for instance

$$y = \frac{1}{x - a}:$$

then, as x keeps increasing continuously from 0 to a value $a-h$, where h is a positive quantity less than any assignable magnitude, it is plain that y also keeps continuously changing through every gradation of value from $-\frac{1}{a}$ to $-\infty$: but when x changes from $a-h$ to $a+h$, y leaps from $-\infty$ to $+\infty$ without passing through the intervening values. Thus we see that in this case y is generally a continuous function, but that it experiences a dissolution of continuity when x becomes equal to a. If we take

$$y = \frac{1}{(x-a)^2},$$

then, although, when $x = a$, y assumes the value ∞, yet this value of x does not correspond to a discontinuous state of the function, since, as x passes from $a-h$ to $a+h$, there is no gap in the range of values of y.

6. Suppose $y = f(x)$ to be the equation to a curve; then, if the function $f(x)$ is continuous for a certain range of values of x, every two points of the locus will be joined by a continuous curve: on the other hand, if there is a dissolution of continuity at any point, and if the function be possible before and after x has passed its critical value, there will be a gap between two points of the curve corresponding to consecutive values of x. Thus, in the instance of the curve

$$y = \frac{1}{x-a},$$

the asymptote, of which the equation is $x = a$, is touched at opposite ends by the curve for two consecutive values of x, one greater and the other less than a by an indefinitely small magnitude.

7. Functions of x, which are expressed by the ordinary signs of algebra and trigonometry, are usually continuous, if we disregard certain dissolutions of continuity corresponding to peculiar and detached values of x. There are however exceptions to this principle. For example, if

$$y = (-a)^x,$$

a being a positive quantity, it is plain that y will be imaginary whenever x is of the form

$$\frac{2\lambda + 1}{2\mu},$$

λ and μ being integers. Now between any two values of x, however little they may differ from each other, we may intercalate an infinite number of fractions of the above form. Thus we see that it is impossible to join by a continuous curve two points of the locus of the equation, corresponding to two systems of real values of x and y, however near they may be to each other. These anomalous functions are inapplicable to questions of natural philosophy, and have attracted but little attention even in pure analysis. In this treatise we shall direct our attention entirely to continuous functions.

8. Functions are distinguished also by the names of *algebraical* and *transcendental*. If y be connected with x by an equation involving only the ordinary operations of addition and subtraction, multiplication and division, evolution and involution of assigned degrees, y is said to be an algebraical function of x. If the equation of connection does not satisfy this condition, involving for instance exponential, logarithmic, or circular functions of x and y, then y is said to be a transcendental function of x. Thus, for examples, in the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1,$$

y is an algebraical function of x; and a transcendental one in the equations

$$a^x + b^y = c^{x+y}, \qquad \log(xy) = x + y,$$
$$x \sin y + y \sin x = \sin(xy).$$

9. It frequently happens that, if

$$F(x, y) = 0$$

be the equation to a curve, y will for a certain value of x experience a dissolution of continuity by becoming impossible, although the curve is itself continuous at the point. Thus if

$$y^2 = 4mx,$$

or $$y = \pm 2m^{\frac{1}{2}}x^{\frac{1}{2}},$$

y will continuously vary from one value to another as x decreases from any assigned positive value down to zero, but, the moment x becomes negative, y becomes impossible. The two branches corresponding to the double sign, each of which terminates abruptly at the origin, join together at this point and thus form a continuous curve.

CHAPTER II.

PRINCIPLES OF DIFFERENTIATION.

SECTION I. GENERAL FUNCTIONS.

Definition of a Differential Coefficient.

10. LET y be a certain function of x, and let y' be the value assumed by y when x becomes x'. Then, as x' keeps continuously approaching to the value of x, the fraction

$$\frac{y'-y}{x'-x}$$

will continuously tend towards a certain value from which it will ultimately differ by a quantity less than any assignable magnitude, or, in other words, to which it will be ultimately equal. The indefinitely small values of the differences $x'-x$, $y'-y$, are usually denoted by the symbols δx, δy, and the ultimate value of the fraction

$$\frac{y'-y}{x'-x} \text{ or } \frac{\delta y}{\delta x}$$

is ordinarily represented by

$$\frac{dy}{dx}.$$

In this expression dx and dy are any quantities whatever, either finite or infinitesimal, which are in the ratio of the ultimate values of δx and δy. The fraction $\frac{dy}{dx}$ is called the *differential coefficient* of y with regard to x, the quantities dx and dy being called the *differentials* of x and y. The object of the Differential Calculus is to investigate the properties of differentials and differential coefficients, and to develop the general principles of their application to the theory of coordinate geometry and other branches of pure mathematics, and to the estimation of the phenomena of nature.

Ex. Let $y = x^3$: then $y' = x'^3$: whence

$$y' - y = x'^3 - x^3 = (x' - x)(x'^2 + x'x + x^2),$$

$$\frac{\delta y}{\delta x} = x'^2 + x'x + x^2.$$

When x' approaches indefinitely near to x, the left-hand member of the equation becomes $\frac{dy}{dx}$, and the right-hand member assumes its limiting value $3x^2$: thus

$$\frac{dy}{dx} = 3x^2,$$

or

$$dy = 3x^2 dx,$$

that is, the differential coefficient of x^3 with respect to x is $3x^2$, and its differential is $3x^2 dx$.

Differentiation of a Constant.

11. If $y = c$, where c denotes any constant quantity, that is, any quantity which does not experience variation in consequence of a variation in the value of x, then

$$\frac{dy}{dx} = 0.$$

For we have

$$y = c, \quad y' = c,$$

whence

$$\delta y = y' - y = 0, \quad \frac{\delta y}{\delta x} = 0,$$

and therefore, proceeding to the limit,

$$\frac{dy}{dx} = 0,$$

or the differential coefficient of a constant quantity is always zero.

Differentiation of the Sum of a Function and a Constant.

12. If $u = y + c$, where y represents any function of x, and c denotes a constant quantity, then

$$\frac{du}{dx} = \frac{dy}{dx}.$$

Let x', y', u', be simultaneous values of x, y, u; then

$$u = y + c, \qquad u' = y' + c,$$

$$\frac{u' - u}{x' - x} = \frac{y' - y}{x' - x}, \qquad \frac{\delta u}{\delta x} = \frac{\delta y}{\delta x}:$$

whence, proceeding to the limit,

$$\frac{du}{dx} = \frac{dy}{dx},$$

or the differential coefficient of the sum of a function and a constant is the same as that of the function alone.

Differentiation of the Product of a Function and a Constant.

13. If $u = cy$, where c represents a constant quantity and y a function of x; then

$$\frac{du}{dx} = c\frac{dy}{dx}.$$

Let x', y', u', be simultaneous values of x, y, u; then

$$u = cy, \qquad u' = cy',$$

$$\frac{u' - u}{x' - x} = c\frac{y' - y}{x' - x}, \qquad \frac{\delta u}{\delta x} = c\frac{\delta y}{\delta x},$$

and therefore, in the limit,

$$\frac{du}{dx} = c\frac{dy}{dx},$$

or the differential coefficient of the product of a function and a constant is equal to the product of the constant and the differential coefficient of the function.

Differentiation of a Sum of Functions.

14. If $u = y_1 + y_2 + y_3 + \dots + y_n$, where $y_1, y_2, y_3, \dots y_n$, denote any functions whatever of x, then

$$\frac{du}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} + \frac{dy_3}{dx} + \dots + \frac{dy_n}{dx}.$$

In fact, taking any simultaneous values $x', u', y'_1, y'_2, y'_3, \dots y'_n$, of $x, u, y_1, y_2, y_3, \dots y_n$, we have

$$u = y_1 + y_2 + y_3 + \dots + y_n,$$
$$u' = y'_1 + y'_2 + y'_3 + \dots + y'_n,$$

whence $$\frac{u'-u}{x'-x} = \frac{y'_1 - y_1}{x'-x} + \frac{y'_2 - y_2}{x'-x} + \frac{y'_3 - y_3}{x'-x} + \dots + \frac{y'_n - y_n}{x'-x},$$

or $$\frac{\delta u}{\delta x} = \frac{\delta y_1}{\delta x} + \frac{\delta y_2}{\delta x} + \frac{\delta y_3}{\delta x} + \dots + \frac{\delta y_n}{\delta x},$$

and therefore, proceeding to the limit,

$$\frac{du}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} + \frac{dy_3}{dx} + \dots + \frac{dy_n}{dx}.$$

Hence the differential coefficient of the sum of any number of functions is equal to the sum of the differential coefficients of the functions taken separately.

Differentiation of the Product of two Functions.

15. If $u = y_1 y_2$, where y_1 and y_2 are any functions of x, then

$$\frac{du}{dx} = y_1 \frac{dy_2}{dx} + y_2 \frac{dy_1}{dx}.$$

We have x', u', y'_1, y'_2, denoting simultaneous values of x, u, y_1, y_2,

$$u = y_1 y_2, \quad u' = y_1' y_2',$$

$$u' - u = y'_1 y'_2 - y_1 y_2 = y_1 (y'_2 - y_2) + y_2 (y'_1 - y_1) + (y'_1 - y_1)(y'_2 - y_2),$$

or $$\frac{\delta u}{\delta x} = y_1 \frac{\delta y_2}{\delta x} + y_2 \frac{\delta y_1}{\delta x} + \frac{\delta y_1 \,.\, \delta y_2}{\delta x}$$

$$= (y_1 + \tfrac{1}{2}\delta y_1) \frac{\delta y_2}{\delta x} + (y_2 + \tfrac{1}{2}\delta y_2) \frac{\delta y_1}{\delta x}:$$

whence, proceeding to the limit, that is, equating δx to zero, and therefore, y_1 and y_2 being supposed to be continuous functions of x, putting also δy_1 and δy_2 each equal to zero, we have

$$\frac{du}{dx} = y_1 \frac{dy_2}{dx} + y_2 \frac{dy_1}{dx},$$

or

$$du = y_1\, dy_2 + y_2\, dy_1.$$

Hence the differential coefficient of the product of two functions is equal to the sum of the products of each function multiplied by the differential coefficient of the other.

Differentiation of the Ratio of two Functions.

16. If $u = \frac{y_1}{y_2}$, then

$$\frac{du}{dx} = \frac{1}{y_2^2}\left(y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx}\right).$$

Taking x', u', y_1', y_2', to denote simultaneous values of x, u, y_1, y_2, we have

$$u' - u = \frac{y_1'}{y_2'} - \frac{y_1}{y_2} = \frac{1}{y_2 y_2'}(y_2 y_1' - y_1 y_2')$$

$$= \frac{1}{y_2 y_2'}\{y_2(y_1' - y_1) - y_1(y_2' - y_2)\},$$

whence

$$\frac{\delta u}{\delta x} = \frac{1}{y_2 y_2'}\left(y_2 \frac{\delta y_1}{\delta x} - y_1 \frac{\delta y_2}{\delta x}\right),$$

and therefore, proceeding to the limit, that is, equating $x' - x$ or δx to zero, we get, observing that y_2' becomes y_2,

$$\frac{du}{dx} = \frac{1}{y_2^2}\left(y_2 \frac{dy_1}{dx} - y_1 \frac{dy_2}{dx}\right),$$

or

$$du = \frac{1}{y_2^2}(y_2 dy_1 - y_1 dy_2).$$

Hence, to differentiate the Ratio of two functions, we have the following rule: Multiply the denominator by the differential coefficient of the numerator, and the numerator by the differential coefficient of the denominator: subtract the latter product from the former: this difference divided by the square of the denominator is the differential coefficient of the Ratio.

Differentiation of the Product of any number of Functions.

17. If $u_n = y_1 . y_2 . y_3 \dots y_n$, the product of n functions of x, then

$$\frac{1}{u_n}\frac{du_n}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} + \frac{1}{y_3}\frac{dy_3}{dx} + \dots + \frac{1}{y_n}\frac{dy_n}{dx}.$$

Since $u_n = u_{n-1} . y_n$, we have, by Art. (15),

$$\frac{du_n}{dx} = y_n \frac{du_{n-1}}{dx} + u_{n-1}\frac{dy_n}{dx},$$

whence $$\frac{1}{u_n}\frac{du_n}{dx} = \frac{1}{u_{n-1}} . \frac{du_{n-1}}{dx} + \frac{1}{y_n}\frac{dy_n}{dx};$$

similarly $$\frac{1}{u_{n-1}}\frac{du_{n-1}}{dx} = \frac{1}{u_{n-2}} . \frac{du_{n-2}}{dx} + \frac{1}{y_{n-1}}\frac{dy_{n-1}}{dx},$$

$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\frac{1}{u_2}\frac{du_2}{dx} = \frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx};$$

adding these equations together, cancelling terms which are common to both sides of the resulting equation, and observing that $u_1 = y_1$, we have

$$\frac{1}{u_n}\frac{du_n}{dx} = \frac{1}{y_1}\frac{dy_1}{dx} + \frac{1}{y_2}\frac{dy_2}{dx} + \frac{1}{y_3}\frac{dy_3}{dx} + \dots + \frac{1}{y_n}\frac{dy_n}{dx},$$

or $$\frac{du_n}{u_n} = \frac{dy_1}{y_1} + \frac{dy_2}{y_2} + \frac{dy_3}{y_3} + \dots + \frac{dy_n}{y_n}.$$

Relation between Inverse Differential Coefficients.

18. If y be a function of x, in which case x will also be a function of y, then

$$\frac{dy}{dx} . \frac{dx}{dy} = 1 .$$

Let x', y', be simultaneous values of x, y; then it is evident that

$$\frac{y'-y}{x'-x}\cdot\frac{x'-x}{y'-y}=1,\qquad \frac{\delta y}{\delta x}\cdot\frac{\delta x}{\delta y}=1;$$

and that, consequently, proceeding to the limit,

$$\frac{dy}{dx}\cdot\frac{dx}{dy}=1.$$

Differentiation of a Function of a Function.

19. If u be a function of y, and y a function of x, then

$$\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}.$$

For we have, by common algebra, taking x', y', u', as simultaneous values of x, y, u,

$$\frac{u'-u}{x'-x}=\frac{u'-u}{y'-y}\cdot\frac{y'-y}{x'-x};$$

and therefore, proceeding to the limit,

$$\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}.$$

COR. If u be a function of y_1, y_1 of y_2, y_2 of y_3,. . . .and y_n of x, it is manifest that we may prove in the same way that

$$\frac{du}{dx}=\frac{du}{dy_1}\cdot\frac{dy_1}{dy_2}\cdot\frac{dy_2}{dy_3}\cdots\cdot\frac{dy_{n-1}}{dy_n}\cdot\frac{dy_n}{dx}.$$

Take for instance three y's; then

$$\frac{u'-u}{x'-x}=\frac{u'-u}{y_1'-y_1}\cdot\frac{y_1'-y_1}{y_2'-y_2}\cdot\frac{y_2'-y_2}{y_3'-y_3}\cdot\frac{y_3'-y_3}{x'-x},$$

and, in the limit,

$$\frac{du}{dx}=\frac{du}{dy_1}\cdot\frac{dy_1}{dy_2}\cdot\frac{dy_2}{dy_3}\cdot\frac{dy_3}{dx}.$$

Differentiation of a Function of two Functions.

20. Let $u=f(y_1, y_2)$, where $f(y_1, y_2)$ denotes any function whatever of y_1 and y_2, each of the quantities y_1 and y_2 being a

function of a third quantity x. Let y_1, y_2, u, become y_1', y_2', u', when x becomes x': then

$$u = f(y_1, y_2), \quad u' = f(y_1', y_2'),$$

$$\begin{aligned} u' - u &= f(y_1', y_2') - f(y_1, y_2) \\ &= f(y_1', y_2) - f(y_1, y_2) + f(y_1', y_2') - f(y_1', y_2), \end{aligned}$$

$$\frac{u' - u}{x' - x} = \frac{f(y_1', y_2) - f(y_1, y_2)}{y_1' - y_1} \cdot \frac{y_1' - y_1}{x' - x} + \frac{f(y_1', y_2') - f(y_1', y_2)}{y_2' - y_2} \cdot \frac{y_2' - y_2}{x' - x}.$$

Now in the limit, when x' differs from x less than by any assignable magnitude,

$$\frac{u' - u}{x' - x} = \frac{Du}{dx}, \quad \frac{y_1' - y_1}{x' - x} = \frac{dy_1}{dx}, \quad \frac{y_2' - y_2}{x' - x} = \frac{dy_2}{dx},$$

$$\frac{f(y_1', y_2) - f(y_1, y_2)}{y_1' - y_1} = \frac{df(y_1, y_2)}{dy_1} = \frac{du}{dy_1},$$

and, first replacing y_1' by y_1, and then y_2' by y_2,

$$\frac{f(y_1', y_2') - f(y_1', y_2)}{y_2' - y_2} = \frac{f(y_1, y_2') - f(y_1, y_2)}{y_2' - y_2} = \frac{df(y_1, y_2)}{dy_2} = \frac{du}{dy_2};$$

hence
$$\frac{Du}{dx} = \frac{du}{dy_1} \cdot \frac{dy_1}{dx} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx} \quad \ldots\ldots\ldots\ldots \quad (1).$$

In this equation it is very important to observe that the numerators of the two fractions

$$\frac{du}{dy_1}, \quad \frac{du}{dy_2},$$

although represented by the same symbol du, are essentially different, the numerator of the former corresponding to the ultimate value of the increment

$$f(y_1', y_2) - f(y_1, y_2),$$

and the numerator of the latter to the ultimate value of the increment

$$f(y_1, y_2') - f(y_1, y_2).$$

The numerator of the fraction

$$\frac{Du}{dx},$$

which corresponds to the ultimate value of the increment

$$f(y_1', y_2') - f(y_1, y_2),$$

and which we have represented by a distinct symbol Du, is evidently different from either of the numerators of the fractions $\frac{du}{dy_1}$, $\frac{du}{dy_2}$.

In order to obviate all possibility of confusion, we might use the symbols $d_{y_1}u$, $d_{y_2}u$, to denote the numerators of $\frac{du}{dy_1}$, $\frac{du}{dy_2}$, the suffixes serving to point out the origin of the two differentials. Such a notation, however, although remarkably clear, would frequently be very embarrassing, especially in long operations. It will be sufficient for distinctness if we remember to regard $\frac{du}{dy_1}$ and $\frac{du}{dy_2}$ as fractions the denominators of which are inseparably attached to the numerators, the symbols dy_1 and dy_2, which express the denominators, thus serving to indicate the true nature of the du in the numerators.

If, however, as will be sometimes convenient, we do put $d_{y_1}u$, $d_{y_2}u$, instead of du, in the expressions $\frac{du}{dy_1}$, $\frac{du}{dy_2}$, we shall then be at liberty to treat these differential coefficients as ordinary algebraical fractions: thus

$$\frac{Du}{dx} = \frac{d_{y_1}u}{dy_1} \cdot \frac{dy_1}{dx} + \frac{d_{y_2}u}{dy_2} \cdot \frac{dy_2}{dx} \ldots\ldots\ldots\ldots (2),$$

may be written, multiplying both sides of the equation by dx,

$$Du = d_{y_1}u + d_{y_2}u \ldots\ldots\ldots\ldots (3).$$

The quantity $d_{y_1}u$ denotes the differential of u taken with regard to y_1, as if y_2 were constant, $d_{y_2}u$ the differential of u taken with regard to y_2 as if y_1 were constant, and Du the differential of u due to the simultaneous variations of y_1 and y_2 dependent upon the variation of x. The quantities $d_{y_1}u$, $d_{y_2}u$, are called the *partial differentials* of u *with regard to* y_1, y_2, respectively, and Du its *total differential.* The quantities

$$\frac{d_{y_1}u}{dy_1}, \quad \frac{d_{y_2}u}{dy_2},$$

in the equation (2), or their equivalents

$$\frac{du}{dy_1}, \ \frac{du}{dy_2},$$

in the equation (1), are called the *partial differential coefficients* of u *with regard to* y_1, y_2, respectively. Finally, $\frac{Du}{dx}$ is called the *total differential coefficient* of u.

The equation (3) shews that the *total differential* of u is equal to the sum of its *partial differentials*.

Differentiation of a Function of any number of Functions of a single Variable.

21. Let $u = f(y_1, y_2, y_3)$, a function of three variables y_1, y_2, y_3, each of which is a function of x. Then if u', y_1', y_2', y_3', be simultaneous values of u, y_1, y_2, y_3, we have

$$u = f(y_1, y_2, y_3), \quad u' = f(y_1', y_2', y_3'),$$

$$u' - u = f(y_1', y_2', y_3') - f(y_1, y_2, y_3),$$

$$\frac{u' - u}{x' - x} = \frac{f(y_1', y_2, y_3) - f(y_1, y_2, y_3)}{y_1' - y_1} \cdot \frac{y_1' - y_1}{x' - x}$$
$$+ \frac{f(y_1', y_2', y_3) - f(y_1', y_2, y_3)}{y_2' - y_2} \cdot \frac{y_2' - y_2}{x' - x}$$
$$+ \frac{f(y_1', y_2', y_3') - f(y_1', y_2', y_3)}{y_3' - y_3} \cdot \frac{y_3' - y_3}{x' - x}.$$

Proceeding to the limit we get, as in the case of two functions,

$$\frac{Du}{dx} = \frac{du}{dy_1} \cdot \frac{dy_1}{dx} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx} + \frac{du}{dy_3} \cdot \frac{dy_3}{dx} \ \ldots\ldots(1).$$

In this equation $\frac{du}{dy_1}$, $\frac{du}{dy_2}$, $\frac{du}{dy_3}$, are the partial differential coefficients of u with regard to y_1, y_2, y_3, respectively; and $\frac{Du}{dx}$ the total differential coefficient of u with regard to x. If we adopt the suffix notation, the equation (1) may be written

$$\frac{Du}{dx} = \frac{d_{y_1}u}{dy_1} \cdot \frac{dy_1}{dx} + \frac{d_{y_2}u}{dy_2} \cdot \frac{dy_2}{dx} + \frac{d_{y_3}u}{dy_3} \cdot \frac{dy_3}{dx} \ \ldots\ldots(2).$$

Multiplying the equation (2) by dx, and putting

$$\frac{d_{y_1}u}{dy_1} \cdot dy_1 = d_{y_1}u, \quad \frac{d_{y_2}u}{dy_2} \cdot dy_2 = d_{y_2}u, \quad \frac{d_{y_3}u}{dy_3} \cdot dy_3 = d_{y_3}u,$$

we have

$$Du = d_{y_1}u + d_{y_2}u + d_{y_3}u \dots\dots\dots\dots (3).$$

The same theorem may evidently be extended to any number of functions; so that, if

$$u = f(y_1, y_2, y_3, \dots y_n)$$

$y_1, y_2, y_3, \dots y_n$, being any n functions of x, then

$$\frac{Du}{dx} = \frac{du}{dy_1} \cdot \frac{dy_1}{dx} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx} + \frac{du}{dy_3} \cdot \frac{dy_3}{dx} + \dots + \frac{du}{dy_n} \cdot \frac{dy_n}{dx},$$

or, replacing du in the expressions

$$\frac{du}{dy_1}, \quad \frac{du}{dy_2}, \quad \frac{du}{dy_3}, \dots \quad \frac{du}{dy_n},$$

by $d_{y_1}u$, $d_{y_2}u$, $d_{y_3}u$, ... $d_{y_n}u$ respectively, and clearing the equation of fractional forms,

$$Du = d_{y_1}u + d_{y_2}u + d_{y_3}u + \dots + d_{y_n}u.$$

It may therefore be stated as a general proposition, that the total differential of a function of any number of functions of a variable is equal to the sum of its partial differentials taken on the hypothesis of the separate variation of each of the several subordinate functions of the variable.

Cor. If any one of the quantities $y_1, y_2, y_3, \dots y_n$, y_1 for instance, be equal to x, which is the most simple form of functionality, then from the above demonstration it is plain that we may replace the corresponding term

$$\frac{du}{dy_1} \cdot \frac{dy_1}{dx} \text{ by } \frac{du}{dx}:$$

thus if

$$u = f(x, y_2, y_3, y_4, \dots y_n),$$

then

$$\frac{Du}{dx} = \frac{du}{dx} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx} + \frac{du}{dy_3} \cdot \frac{dy_3}{dx} + \dots + \frac{du}{dy_n} \cdot \frac{dy_n}{dx},$$

or, transforming the equation from differential coefficients to differentials,

$$Du = d_x u + d_{y_2}u + d_{y_3}u + \dots\dots + d_{y_n}u.$$

Differentiation of an implicit Function of a single Variable.

22. Let $$v = \phi(x, y) = 0,$$

y being therefore an implicit function of x: then v', x', y', being corresponding values of v, x, y,

$$v = \phi(x, y) = 0, \quad v' = \phi(x', y') = 0,$$

$$v' - v = \phi(x', y') - \phi(x, y) = 0,$$

or

$$\frac{v' - v}{x' - x} = \frac{\phi(x', y) - \phi(x, y)}{x' - x} + \frac{\phi(x', y') - \phi(x', y)}{y' - y} \cdot \frac{y' - y}{x' - x} = 0 \ldots (1).$$

Now in the limit, when x' approaches indefinitely near to x, and therefore y' to y, we have

$$\frac{y' - y}{x' - x} = \frac{dy}{dx}, \quad \frac{v' - v}{x' - x} = \frac{Dv}{dx},$$

$$\frac{\phi(x', y) - \phi(x, y)}{x' - x} = \frac{d\phi(x,y)}{dx} = \frac{dv}{dx},$$

and, first replacing x' by x, and then y' by y,

$$\frac{\phi(x', y') - \phi(x', y)}{y' - y} = \frac{\phi(x, y') - \phi(x, y)}{y' - y} = \frac{d\phi(x, y)}{dy} = \frac{dv}{dy}:$$

hence the equation (1) becomes

$$\frac{Dv}{dx} = \frac{dv}{dx} + \frac{dv}{dy} \cdot \frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots (2).$$

This result gives us the expression for $\frac{dy}{dx}$ in terms of the partial differential coefficients of v with regard to x and y taken successively as separately varying. If we replace the symbol dv in the numerators of the fractions

$$\frac{dv}{dx}, \quad \frac{dv}{dy},$$

by the expressive forms $d_x v$, $d_y v$, we have, transforming the equation (2) from differential coefficients to differentials,

$$Dv = d_x v + d_y v = 0.$$

This result shews that if any function of x and y be always zero, its total differential or the sum of its partial differentials is always zero.

General Theory of the Differentiation of implicit Functions of a single Variable.

23. If $v_1 = 0,\ v_2 = 0,\ v_3 = 0,\ \ldots\ldots v_n = 0$,

where $v_1, v_2, v_3, \ldots\ldots v_n$, are n functions of $x, y_1, y_2, y_3, \ldots\ldots y_n$, the variables $y_1, y_2, y_3, \ldots\ldots y_n$, being therefore implicit functions of x, then

$$\frac{dv_1}{dx} + \frac{dv_1}{dy_1} \cdot \frac{dy_1}{dx} + \frac{dv_1}{dy_2} \cdot \frac{dy_2}{dx} + \frac{dv_1}{dy_3} \cdot \frac{dy_3}{dx} + \ldots\ldots + \frac{dv_1}{dy_n} \cdot \frac{dy_n}{dx} = 0,$$

together with $n-1$ additional equations involving $v_2, v_3, v_4 \cdots v_n$, precisely as v_1 is involved in this.

Let $x', y_1', y_2', y_3', \ldots\ldots y_n', v_1'$, be corresponding values of $x, y_1, y_2, y_3, \ldots\ldots y_n, v_1$; then

$$v_1 = f(x, y_1, y_2, y_3, \ldots\ldots y_n) = 0, \quad v_1' = f(x', y_1', y_2', y_3', \ldots y_n') = 0;$$

hence

$$\begin{aligned} 0 = v_1' - v_1 &= f(x', y_1', y_2', y_3', \ldots\ldots y_n') - f(x, y_1, y_2, y_3, \ldots\ldots y_n) \\ &= f(x', y_1, y_2, y_3, \ldots\ldots y_n) - f(x, y_1, y_2, y_3, \ldots\ldots y_n) \\ &+ f(x', y_1', y_2, y_3, \ldots\ldots y_n) - f(x', y_1, y_2, y_3, \ldots\ldots y_n) \\ &+ f(x', y_1', y_2', y_3, \ldots\ldots y_n) - f(x', y_1', y_2, y_3, \ldots\ldots y_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ f(x', y_1', y_2', y_3' \ldots\ldots y_n') - f(x', y_1', y_2', y_3', \ldots\ldots y_n): \end{aligned}$$

whence, $\delta_x, \delta_{y1}, \delta_{y_2}, \ldots\ldots \delta_{y_n}$, denoting the partial increments of the functions to which they are prefixed with regard to $x, y_1, y_2, \ldots\ldots y_n$, respectively, and Δ the total increment of v_1, we have

$$\begin{aligned} 0 = \frac{\Delta v_1}{\delta x} = \frac{\delta_x f(x, y_1, y_2, y_3, \cdots y_n)}{\delta x} &+ \frac{\delta_{y1} f(x', y_1, y_2, y_3, \cdots y_n)}{\delta y_1} \cdot \frac{\delta y_1}{\delta x} \\ &+ \frac{\delta_{y_2} f(x', y_1', y_2, y_3, \cdots y_n)}{\delta y_2} \cdot \frac{\delta y_2}{\delta x} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \frac{\delta_{y_n} f(x', y_1', y_2', y_3' \cdots y'_{n-1}, y_n)}{\delta y_n} \cdot \frac{\delta y_n}{\delta x}. \end{aligned}$$

Proceeding to the limit, we have, $x, y_1, y_2, y_3, \ldots y_n$, being the ultimate values of $x', y_1', y_2', y_3', \ldots y_n'$,

$$0 = \frac{Dv_1}{dx} = \frac{dv_1}{dx} + \frac{dv_1}{dy_1} \cdot \frac{dy_1}{dx} + \frac{dv_1}{dy_2} \cdot \frac{dy_2}{dx} + \frac{dv_1}{dy_3} \cdot \frac{dy_3}{dx} + \ldots$$
$$+ \frac{dv_1}{dy_n} \cdot \frac{dy_n}{dx} \ldots (1).$$

The analogous equations in regard to $v_2, v_3, v_4, \ldots v_n$, may be established in the same way.

Multiplying the equation (1) by dx, we have

$$0 = Dv_1 = \frac{dv_1}{dx} \cdot dx + \frac{dv_1}{dy_1} \cdot dy_1 + \frac{dv_1}{dy_2} \cdot dy_2 + \frac{dv_1}{dy_3} \cdot dy_3 + \ldots + \frac{dv_1}{dy_n} \cdot dy_n,$$

or, if we express the different partial differentials of v_1 by suggestive suffixes,

$$0 = Dv_1 = d_x v_1 + d_{y_1} v_1 + d_{y_2} v_1 + d_{y_3} v_1 + \ldots + d_{y_n} v_1.$$

From the equation (1) together with the $(n - 1)$ analogous equations, making in all n linear equations, we may determine the n differential coefficients

$$\frac{dy_1}{dx}, \frac{dy_2}{dx}, \frac{dy_3}{dx}, \ldots \frac{dy_n}{dx},$$

in terms of the $n(n + 1)$ partial differential coefficients of $v_1, v_2, v_3, \ldots v_n$.

Total Differentiation of a Function of Functions of independent Variables.

24. We have now fully considered the principle of differentiating a function of functions, the subordinate functions being dependent each of them upon one and the same variable. Suppose, however, that

$$u = f(y_1, y_2, y_3, \ldots y_n),$$

and that $y_1, y_2, y_3, \ldots y_n$, are not all of them dependent upon a single variable, but that they are functions of several independent variables. Let $u', y_1', y_2', y_3', \ldots y_n'$, be corresponding values of $u, y_1, y_2, y_3, \ldots y_n$: then

$$\begin{aligned} u' - u &= f(y_1', y_2', y_3', \ldots . y_n') - f(y_1, y_2, y_3, \ldots . y_n) \\ &= f(y_1', y_2, y_3, \ldots . y_n) - f(y_1, y_2, y_3, \ldots y_n) \\ &+ f(y_1', y_2', y_3, \ldots . y_n) - f(y_1', y_2, y_3, \ldots . y_n) \\ &+ f(y_1', y_2', y_3', \ldots . y_n) - f(y_1', y_2', y_3, \ldots . y_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ f(y_1', y_2', y_3', \ldots y'_{n-1}, y_n') - f(y_1', y_2', y_3', \ldots y'_{n-1}, y_n), \end{aligned}$$

or

$$\begin{aligned} \Delta u = \delta_{y_1} f(y_1, y_2, y_3, \ldots y_n) + \delta_{y_2} f(y_1', y_2, y_3, \ldots y_n) + \delta_{y_3} f(y_1', y_2', y_3, \ldots y_n) \\ + \ldots . + \delta_n f(y_1', y_2', y_3', \ldots . y'_{n-1}, y_n). \end{aligned}$$

Now when these increments are diminished indefinitely by the continuous approximation of y_1', y_2', $y_3', \ldots . y_n'$, to the values $y_1, y_2, y_3, \ldots . y_n$, and in correspondency with any restrictions to which these variables may be subject, let quantities finite or infinitesimal which are proportional to these vanishing increments be called the *differentials* of the corresponding incremented quantities: this definition of differentials is merely an extension of the definition for the case of one independent variable to that of any number. Then, by the notation of differentials, we have

$$Du = d_{y_1} u + d_{y_2} u + d_{y_3} u + \ldots . + d_{y_n} u.$$

Ex. Suppose that

$$u = f(y_1, y_2, y_3) = y_1 + y_2 + y_3,$$

$$y_1 = x_1 + x_2, \quad y_2 = x_1^2 + x_2^2, \quad y_3 = x_1^3 + x_2^3;$$

then

$$\delta_{y_1} u = \delta y_1, \quad \delta_{y_2} u = \delta y_2, \quad \delta_{y_3} u = \delta y_3,$$

$$d_{y_1} u = dy_1, \quad d_{y_2} u = dy_2, \quad d_{y_3} u = dy_3,$$

$$Du = dy_1 + dy_2 + dy_3.$$

$$\begin{aligned} \delta y_1 = \delta x_1 + \delta x_2, \quad \delta y_2 &= x_1'^2 - x_1^2 + x_2'^2 - x_2^2 \\ &= (x_1' + x_1)\, \delta x_1 + (x_2' + x_2)\, \delta x_2, \end{aligned}$$

$$\begin{aligned} \delta y_3 &= x_1'^3 - x_1^3 + x_2'^3 - x_2^3 \\ &= (x_1'^2 + x_1' x_1 + x_1^2)\, \delta x_1 + (x_2'^2 + x_2' x_2 + x_2^2)\, \delta x_2 : \end{aligned}$$

hence, by the definition,

$$dy_1 = dx_1 + dx_2, \quad dy_2 = 2x_1\, dx_1 + 2x_2 dx_2, \quad dy_3 = 3x_1^2\, dx_1 + 3x_2^2\, dx_2;$$

$$\begin{aligned} Du &= dx_1 + dx_2 + 2x_1 dx_1 + 2x_2 dx_2 + 3x_1^2 dx_1 + 3x_2^2\, dx_2 \\ &= (1 + 2x_1 + 3x_1^2)\, dx_1 + (1 + 2x_2 + 3x_2^2)\, dx_2. \end{aligned}$$

Partial Differentiation of an explicit Function of three Variables one of which is a Function of the other two.

25. Suppose that $u = f(x, y, z)$,

z being some function of two independent variables x and y.

Since x and y are supposed to vary independently of each other, the variation of z being dependent upon the variations of x and y, we may assume y to remain unchanged while x and therefore z varies: then, the expression

$$\frac{Du}{dx}$$

being taken to denote the total differential coefficient of u, as far as u is affected, both immediately by the variation of x and indirectly by the variation of z as consequent upon that of x, we have, by Art. (21), Cor.,

$$\frac{Du}{dx} = \frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} \quad \ldots\ldots\ldots\ldots \quad (1).$$

In like manner, y being supposed variable and x constant,

$$\frac{Du}{dy} = \frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy} \quad \ldots\ldots\ldots\ldots \quad (2).$$

In these equations $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are the partial differential coefficients of z with regard to x and y respectively; $\frac{du}{dx}$ and $\frac{du}{dy}$ are the partial differential coefficients of u with regard to x and y. In the equation (1), $\frac{du}{dz}$ represents the value of the ultimate ratio of the increment of u to the increment of z, when z receives an increment in consequence of the variation of x: in the equation (2), $\frac{du}{dz}$ represents the value of the ultimate ratio of the increment of u to the increment of z, when z receives an increment in consequence of the variation of y. It is important however to observe that in both cases the actual value of $\frac{du}{dz}$ must be the same, the origin of the variation of z evidently not affecting the ultimate ratio in question. We are at liberty therefore to consider dz as the total differential of z in the denominator of $\frac{du}{dz}$ in both equations, the value of du being accordingly also the same in both. The equations written in the most expressive form would accordingly be

$$\frac{D_x u}{dx} = \frac{d_x u}{dx} + \frac{d_z u}{Dz} \cdot \frac{d_x z}{dx} \quad \ldots\ldots\ldots\ldots (3),$$

$$\frac{D_y u}{dy} = \frac{d_y u}{dy} + \frac{d_z u}{Dz} \cdot \frac{d_y z}{dy} \quad \ldots\ldots\ldots\ldots (4).$$

Owing to the complexity of the notation in (3) and (4), it will be desirable to adhere to the form of expression which we have given in (1) and (2). No danger of confusion can arise from the several meanings of Du, du, dz, provided that we remember to regard as monads the expressions

$$\frac{Du}{dx}, \ \frac{Du}{dy}, \ \frac{du}{dx}, \ \frac{du}{dy}, \ \frac{du}{dz}, \ \frac{dz}{dx}, \ \frac{dz}{dy},$$

the denominators of these indissoluble fractions sufficing to suggest the significations of their numerators.

Multiplying (3) and (4) by dx and dy respectively, and adding, we have

$$D_x u + D_y u = d_x u + d_y u + \frac{d_z u}{Dz} \cdot (d_x z + d_y z):$$

but, by Art. (24), $\qquad Dz = d_x z + d_y z;$

hence $\qquad D_x u + D_y u = d_x u + d_y u + d_z u:$

but, by Art. (24), we have also

$$Du = d_x u + d_y u + d_z u;$$

hence $\qquad Du = D_x u + D_y u = d_x u + d_y u + d_z u.$

Partial Differentiation of an explicit Function of n + r *Variables,* r *independent and* n *dependent.*

26. Let $u = f(x_1, x_2, x_3, \ldots x_r, y_1, y_2, y_3, \ldots y_n)$, where $y_1, y_2, y_3, \ldots y_n$, are each of them functions of r independent variables $x_1, x_2, x_3, \ldots x_r$.

Then, differentiating successively with regard to $x_1, x_2, x_3, \dots x_r$, each of these quantities being taken in turn as the only variable among them, we have, by Art. (21),

$$\frac{Du}{dx_1} = \frac{du}{dx_1} + \frac{du}{dy_1} \cdot \frac{dy_1}{dx_1} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx_1} + \frac{du}{dy_3} \cdot \frac{dy_3}{dx_1} + \dots + \frac{du}{dy_n} \cdot \frac{dy_n}{dx_1},$$

$$\frac{Du}{dx_2} = \frac{du}{dx_2} + \frac{du}{dy_1} \cdot \frac{dy_1}{dx_2} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx_2} + \frac{du}{dy_3} \cdot \frac{dy_3}{dx_2} + \dots + \frac{du}{dy_n} \cdot \frac{dy_n}{dx_2},$$

$$\frac{Du}{dx_3} = \frac{du}{dx_3} + \frac{du}{dy_1} \cdot \frac{dy_1}{dx_3} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx_3} + \frac{du}{dy_3} \cdot \frac{dy_3}{dx_3} + \dots + \frac{du}{dy_n} \cdot \frac{dy_n}{dx_3},$$

....

....

$$\frac{Du}{dx_r} = \frac{du}{dx_r} + \frac{du}{dy_1} \cdot \frac{dy_1}{dx_r} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx_r} + \frac{du}{dy_3} \cdot \frac{dy_3}{dx_r} + \dots + \frac{du}{dy_n} \cdot \frac{dy_n}{dx_r}.$$

Partial Differentiation of an implicit Function of two independent Variables.

27. Let z be an implicit function of two independent variables x and y by virtue of an equation

$$u = f(x, y, z) = 0.$$

Then supposing, as we are evidently at liberty to do, that y remains constant while x and consequently z varies, we have, by Art. (22),

$$\frac{Du}{dx} = \frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} = 0.$$

Again, supposing y variable and x constant, we shall have also

$$\frac{Du}{dy} = \frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy} = 0.$$

Partial Differentiation of implicit Functions of any number of independent Variables.

28. Let $\quad v_1 = 0, \quad v_2 = 0, \quad v_3 = 0, \dots v_n = 0,$
where $v_1, v_2, v_3, \dots v_n$, are n functions of $n + r$ variables $x_1, x_2, x_3, \dots x_r, y_1, y_2, y_3, \dots y_n$: then each of the variables $y_1, y_2, y_3, \dots y_n$,

may be regarded as a function of r independent variables $x_1, x_2, x_3, \ldots . x_r$. We are therefore at liberty to consider $x_2, x_3, \ldots . x_r$, as all constant, and to regard $y_1, y_2, y_3 \ldots . y_n$, as functions of a variable x_1. We have therefore, by Art. (23),

$$0 = \frac{Dv_1}{dx_1} = \frac{dv_1}{dx_1} + \frac{dv_1}{dy_1} \cdot \frac{dy_1}{dx_1} + \frac{dv_1}{dy_2} \cdot \frac{dy_2}{dx_1} + \frac{dv_1}{dy_3} \cdot \frac{dy_3}{dx_1} + \ldots + \frac{dv_1}{dy_n} \cdot \frac{dy_n}{dx_1}.$$

Similarly, $x_2, x_3, \ldots . x_r$, successively taking the place of x_1,

$$0 = \frac{Dv_1}{dx_2} = \frac{dv_1}{dx_2} + \frac{dv_1}{dy_1} \cdot \frac{dy_1}{dx_2} + \frac{dv_1}{dy_2} \cdot \frac{dy_2}{dx_2} + \frac{dv_1}{dy_3} \cdot \frac{dy_3}{dx_2} + \ldots + \frac{dv_1}{dy_n} \cdot \frac{dy_n}{dx_2},$$

$$0 = \frac{Dv_1}{dx_3} = \frac{dv_1}{dx_3} + \frac{dv_1}{dy_1} \cdot \frac{dy_1}{dx_3} + \frac{dv_1}{dy_2} \cdot \frac{dy_2}{dx_3} + \frac{dv_1}{dy_3} \cdot \frac{dy_3}{dx_3} + \ldots + \frac{dv_1}{dy_n} \cdot \frac{dy_n}{dx_3},$$

. . . .

. . . .

$$0 = \frac{Dv_1}{dx_r} = \frac{dv_1}{dx_r} + \frac{dv_1}{dy_1} \cdot \frac{dy_1}{dx_r} + \frac{dv_1}{dy_2} \cdot \frac{dy_2}{dx_r} + \frac{dv_1}{dy_3} \cdot \frac{dy_3}{dx_r} + \ldots + \frac{dv_1}{dy_n} \cdot \frac{dy_n}{dx_r}.$$

There will evidently be also r analogous equations in relation to each of the functions $v_2, v_3, v_4, \ldots . v_n$. We thus have nr differential equations, and may thence determine expressions for the nr partial differential coefficients of the dependent variables $y_1, y_2, y_3, \ldots . y_n$, viz.

$$\begin{array}{llll}
\frac{dy_1}{dx_1}, & \frac{dy_2}{dx_1}, & \frac{dy_3}{dx_1}, \ldots\ldots & \frac{dy_n}{dx_1}, \\
\frac{dy_1}{dx_2}, & \frac{dy_2}{dx_2}, & \frac{dy_3}{dx_2}, \ldots\ldots & \frac{dy_n}{dx_2}, \\
\frac{dy_1}{dx_3}, & \frac{dy_2}{dx_3}, & \frac{dy_3}{dx_3}, \ldots\ldots & \frac{dy_n}{dx_3}, \\
\cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\
\cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\
\frac{dy_1}{dx_r}, & \frac{dy_2}{dx_r}, & \frac{dy_3}{dx_r}, \ldots\ldots & \frac{dy_n}{dx_r},
\end{array}$$

in terms of the $n(n+r)$ partial differential coefficients of $v_1, v_2, v_3, \ldots v_n$, taken with regard to the variables $x_1, x_2, x_3, \ldots x_r$, $y_1, y_2, y_3, \cdots y_n$.

SECTION II. SIMPLE FUNCTIONS.

29. In the preceding section we have shewn how to reduce the differentiation of a function of functions to that of the differentiation of its subordinate functions. In this section we shall investigate the differentials of what may be called simple functions, as being the constituent elements or subordinate functions of all the complex functions of algebra. The essential characteristic of a simple function consists in its not being susceptible of resolution into elements more simple than itself, except by the aid of infinite series: the number of simple functions might therefore, as may easily be imagined, be multiplied indefinitely. The algebraical expressions ordinarily adopted as simple functions are the following:

x^m, m being any real quantity whatever,

a^x, $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$,

and the inverse functions

$\log_a x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$.

These expressions have been selected as the elementary functions of ordinary analysis, in consequence of their peculiar utility in the various applications of the science.

To find the Differential Coefficient of x^n *with respect to* x, n *being any rational quantity whatever.*

30. Put $y = x^n$; then, x', y', being corresponding values of x and y, we have

$$y' - y = x'^n - x^n,$$

$$\frac{\delta y}{\delta x} = \frac{x'^n - x^n}{x' - x} = x^{n-1} \cdot \frac{z^n - 1}{z - 1},$$

z being such a quantity that $x' = xz$.

Our object is now to find the limiting value of the fraction

$$\frac{z^n - 1}{z - 1},$$

when z approaches indefinitely near to unity. Now whatever be the value of n, positive, integral, fractional, or negative, we may always express it under the form

$$\frac{p-q}{r},$$

where p, q, r, are positive integers. Hence

$$\frac{z^n-1}{z-1}=\frac{z^{\frac{p-q}{r}}-1}{z-1}=\frac{v^{p-q}-1}{v^r-1}, \text{ putting } z=v^r,$$

$$=\frac{1}{v^q}\cdot\frac{(v^p-1)-(v^q-1)}{v^r-1}.$$

Hence, dividing v^p-1, v^q-1, v^r-1, by $v-1$, observing that p, q, r, are positive integers, we have

$$\frac{z^n-1}{z-1}=\frac{1}{v^q}\cdot\frac{(1+v+v^2+\ldots\ldots+v^{p-1})-(1+v+v^2+\ldots\ldots+v^{q-1})}{1+v+v^2+\ldots\ldots+v^{r-1}}.$$

Now, by making v approach more nearly to unity than by any assignable difference, z will also be made to do so: hence

$$\text{limit of } \frac{z^n-1}{z-1}=\frac{p-q}{r}=n.$$

Hence we see that $\dfrac{dy}{dx}=nx^{n-1}$.

To find the Differential Coefficient of $\log_a x$ *with regard to* x.

31. Put $y=\log_a x$;

then $y'=\log_a x'$,

$$\delta y=\log_a\frac{x'}{x}=\log_a\left(1+\frac{\delta x}{x}\right),$$

$$\frac{\delta y}{\delta x}=\frac{\log_a\left(1+\frac{\delta x}{x}\right)}{\delta x}.$$

Put $\dfrac{\delta x}{x}=\theta$; then

$$\frac{\delta y}{\delta x}=\frac{1}{x}\cdot\frac{\log_a(1+\theta)}{\theta}=\frac{1}{x}\log_a(1+\theta)^{\frac{1}{\theta}}$$

$$=\frac{1}{x}\log_a\left(1+\frac{1}{n}\right)^n, \text{ putting } \theta=\frac{1}{n}.$$

Our object is to determine the value assumed by the expression

$$\log_a\left(1+\frac{1}{n}\right)^n,$$

when $n = \infty$, a value of n consequent upon the evanescent state of δx. Now whether n be a continuous or a discontinuous variable, yet, provided that it become greater than any assignable magnitude, δx will become less than any assignable magnitude, which is the only condition to be fulfilled by δx in the ultimate state of the hypothesis. We will assume then n to represent a positive integer, and proceed to ascertain the limiting value of the function

$$\log_a\left(1+\frac{1}{n}\right)^n,$$

when the integer n becomes great without limit.

By the binomial theorem we have the following expansion, consisting of $n + 1$ terms, for $\left(1+\frac{1}{n}\right)^n$, viz.

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n = 1 + \frac{n}{1}\cdot\frac{1}{n} + \frac{n(n-1)}{1.2}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{n^3} + \dots\\ + \frac{n(n-1)(n-2)\dots(n-\nu+1)}{1.2.3\dots\nu}\cdot\frac{1}{n^\nu} + \dots\\ + \frac{n(n-1)(n-2)\dots(n-\overline{n-1})}{1.2.3\dots n}\cdot\frac{1}{n^n},\end{aligned}$$

$$\begin{aligned}\text{or } \left(1+\frac{1}{n}\right)^n = 1 + \frac{1}{1} + \frac{1}{1.2}\left(1-\frac{1}{n}\right) + \frac{1}{1.2.3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots\\ + \frac{1}{1.2.3\dots\nu}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{\nu-1}{n}\right) + \dots\\ + \frac{1}{1.2.3\dots\nu(\nu+1)\dots n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{\nu-1}{n}\right)\dots\left(1-\frac{n-1}{n}\right).\end{aligned}$$

Now we may take n and ν so large that, if we stop the series at the $(\nu + 1)^{\text{th}}$ term, the sum of all the remaining terms will be less than any assignable magnitude. In fact, this sum is less than

$$\frac{1}{1.2.3\dots\nu(\nu+1)} + \frac{1}{1.2.3\dots\nu(\nu+1)(\nu+2)} + \dots + \frac{1}{1.2.3\dots\nu(\nu+1)\dots n},$$

and *a fortiori* than $\frac{1}{2^\nu} + \frac{1}{2^{\nu+1}} + \dots\dots + \frac{1}{2^{n-1}}$,

and *a fortiori* than

$$\frac{1}{2^\nu} + \frac{1}{2^{\nu+1}} + \dots\dots + \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots\dots,$$

which series is equal to $\frac{1}{2^{\nu-1}}$,

and becomes therefore indefinitely small when ν is increased without limit.

If then we take ν indefinitely large, and neglect accordingly all terms after the $(\nu+1)^{\text{th}}$, and if we then take n, which is of course always larger than ν, an indefinitely large number of a higher order of magnitude than ν, so that in fact the ratio of n to ν shall be indefinitely great, we shall have

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \dots\dots\dots + \frac{1}{1.2.3\dots\dots\nu},$$

an approximation true without limit as ν increases without limit; that is, in the limit,

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \dots\dots\dots \text{ ad infinitum}$$

$$= e, \text{ the base of Napier's logarithms.}$$

Hence we conclude that

$$\frac{dy}{dx} = \frac{1}{x} . \log_a e = \frac{1}{x \log_e a}.$$

To find the Differential Coefficient of a

32. Put $y = a^x, \quad y' = a^{x'} = a^{x+\delta x};$

then $y' - y = a^x . (a^{\delta x} - 1),$

$$\frac{\delta y}{\delta x} = a^x . \frac{a^{\delta x} - 1}{\delta x}.$$

Put $a^{\delta x} - 1 = \frac{1}{n}, \quad \delta x \log_e a = \log_e \left(1 + \frac{1}{n}\right);$

then $\frac{\delta y}{\delta x} = a^x . \frac{\log_e a}{n \log_e \left(1+\frac{1}{n}\right)} = a^x . \frac{\log_e a}{\log_e \left(1+\frac{1}{n}\right)^n}.$

Now, to proceed to the limit, putting n = an indefinitely large positive integer, and thereby rendering δx less than any assignable quantity, we have,

$$\frac{dy}{dx} = a^x . \frac{\log_e a}{\log_e e} = a^x . \log_e a.$$

To find the Differential Coefficient of sin x *with regard to* x.

33. Put $$y = \sin x;$$
then $$\delta y = \sin (x + \delta x) - \sin x$$
$$= 2 \sin \frac{\delta x}{2} . \cos \left(x + \frac{\delta x}{2}\right),$$
whence $$\frac{\delta y}{\delta x} = \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} . \cos \left(x + \frac{\delta x}{2}\right).$$

Now by the seventh Lemma of the first section of Newton's Principia we know that the arc and the chord of any curve vanish in a ratio of equality: whence it follows that the ratio between the sine and the circular measure of an angle is ultimately unity. Hence, in the limit,

$$\frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1,$$

and therefore $$\frac{dy}{dx} = \cos x, \quad dy = \cos x . dx.$$

To find the Differential Coefficient of cos x.

34. Put $$y = \cos x, \quad y' = \cos (x + \delta x);$$
then $$\delta y = \cos (x + \delta x) - \cos x$$
$$= - 2 \sin \left(x + \frac{\delta x}{2}\right) . \sin \frac{\delta x}{2},$$
$$\frac{\delta y}{\delta x} = - \sin \left(x + \frac{\delta x}{2}\right) . \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}};$$

whence, in the limit, $\dfrac{dy}{dx} = -\sin x,$

or $dy = -\sin x \,.\, dx.$

To find the Differential Coefficient of tan x.

35. Put $y = \tan x, \quad y' = \tan(x + \delta x);$

then
$$\delta y = y' - y = \tan(x + \delta x) - \tan x$$
$$= \frac{\sin(x + \delta x)\cos x - \sin x \cos(x + \delta x)}{\cos x \cos(x + \delta x)}$$
$$= \frac{\sin \delta x}{\cos x \,.\, \cos(x + \delta x)},$$
$$\frac{\delta y}{\delta x} = \frac{1}{\cos x \cos(x + \delta x)} \,.\, \frac{\sin \delta x}{\delta x}:$$

proceeding to the limit, when
$$\frac{\sin \delta x}{\delta x} = 1,$$

we see that $\dfrac{dy}{dx} = \dfrac{1}{(\cos x)^2} = (\sec x)^2,$

or $dy = (\sec x)^2\, dx.$

To find the Differential Coefficient of cot x.

36. Put $y = \cot x, \quad y + \delta y = \cot(x + \delta x);$

then
$$\delta y = \cot(x + \delta x) - \cot x$$
$$= \frac{\sin x \cos(x + \delta x) - \cos x \,.\, \sin(x + \delta x)}{\sin x \,.\, \sin(x + \delta x)}$$
$$= -\frac{\sin \delta x}{\sin x \,.\, \sin(x + \delta x)},$$
$$\frac{\delta y}{\delta x} = -\frac{1}{\sin x \,.\, \sin(x + \delta x)} \,.\, \frac{\sin \delta x}{\delta x}:$$

proceeding to the limit, since ultimately
$$\frac{\sin \delta x}{\delta x} = 1,$$

we have $$\frac{dy}{dx} = -\frac{1}{(\sin x)^2} = -(\text{cosec } x)^2,$$

or $$dy = -(\text{cosec } x)^2 . dx.$$

To find the Differential Coefficient of sec x.

37. Putting $y = \sec x, \quad y + \delta y = \sec(x + \delta x)$,

we get $$\delta y = \sec(x + \delta x) - \sec x$$

$$= \frac{\cos x - \cos(x + \delta x)}{\cos x . \cos(x + \delta x)}$$

$$= \frac{2 \sin \frac{\delta x}{2} . \sin\left(x + \frac{\delta x}{2}\right)}{\cos x . \cos(x + \delta x)},$$

$$\frac{\delta y}{\delta x} = \frac{\sin\left(x + \frac{\delta x}{2}\right)}{\cos x . \cos(x + \delta x)} . \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}:$$

proceeding to the limit, when

$$\frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1,$$

we have $$\frac{dy}{dx} = \frac{\sin x}{(\cos x)^2} = \tan x . \sec x,$$

or $$dy = \tan x . \sec x . dx.$$

To find the Differential Coefficient of cosec x.

38. Putting $y = \text{cosec } x, \quad y + \delta y = \text{cosec}(x + \delta x)$,

we have $$\delta y = \text{cosec}(x + \delta x) - \text{cosec } x$$

$$= \frac{\sin x - \sin(x + \delta x)}{\sin x . \sin(x + \delta x)}$$

$$= -\frac{2 \sin \frac{\delta x}{2} . \cos\left(x + \frac{\delta x}{2}\right)}{\sin x . \sin(x + \delta x)},$$

$$\frac{\delta y}{\delta x} = -\frac{\cos\left(x + \frac{\delta x}{2}\right)}{\sin x \,.\, \sin(x + \delta x)} \cdot \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}:$$

in the limit $$\frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1,$$

and therefore $$\frac{dy}{dx} = -\frac{\cos x}{(\sin x)^2} = -\cot x \,.\, \operatorname{cosec} x,$$

or $$dy = -\cot x \,.\, \operatorname{cosec} x \,.\, dx.$$

To find the Differential Coefficient of $\sin^{-1}$ x *with regard to* x.

39. Put $y = \sin^{-1} x, \quad y + \delta y = \sin^{-1}(x + \delta x),$

then $$\sin y = x, \quad \sin(y + \delta y) = x + \delta x,$$

$$\sin(y + \delta y) - \sin y = \delta x,$$

or $$2\cos\left(y + \frac{\delta y}{2}\right) . \sin\frac{\delta y}{2} = \delta x;$$

and therefore $$\frac{\delta y}{\delta x} = \frac{\frac{\delta y}{2}}{\sin\frac{\delta y}{2}} \cdot \frac{1}{\cos\left(y + \frac{\delta y}{2}\right)}.$$

Now, proceeding to the limit, when δx and δy assume values less than any assignable magnitudes, we have

$$\frac{\delta y}{\delta x} = \frac{dy}{dx}, \quad \frac{\frac{\delta y}{2}}{\sin\frac{\delta y}{2}} = 1, \quad \cos\left(y + \frac{\delta y}{2}\right) = \cos y:$$

hence $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\{1 - (\sin y)^2\}^{\frac{1}{2}}} = \frac{1}{(1 - x^2)^{\frac{1}{2}}},$$

or $$dy = \frac{dx}{(1 - x^2)^{\frac{1}{2}}}.$$

To find the Differential Coefficient of $\cos^{-1} x$.

40. Put $\quad y = \cos^{-1} x, \quad y + \delta y = \cos^{-1}(x + \delta x),$

whence $\quad \cos y = x, \quad \cos(y + \delta y) = x + \delta x,$

$$\cos(y + \delta y) - \cos y = \delta x,$$

or $$-2 \sin\left(y + \frac{\delta y}{2}\right) . \sin\frac{\delta y}{2} = \delta x,$$

whence $$\frac{\delta y}{\delta x} = -\frac{\frac{\delta y}{2}}{\sin\frac{\delta y}{2}} . \frac{1}{\sin\left(y + \frac{\delta y}{2}\right)}:$$

proceeding to the limit, we have

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\{1 - (\cos y)^2\}^{\frac{1}{2}}} = -\frac{1}{(1 - x^2)^{\frac{1}{2}}},$$

or $$dy = -\frac{dx}{(1 - x^2)^{\frac{1}{2}}}.$$

To find the Differential Coefficient of $\tan^{-1} x$.

41. Proceeding in the same way as in the investigation of the differentials of $\sin^{-1} x$ and $\cos^{-1} x$, we have

$$y = \tan^{-1} x, \quad y + \delta y = \tan^{-1}(x + \delta x),$$

$$\tan y = x, \quad \tan(y + \delta y) = x + \delta x,$$

$$\delta x = \tan(y + \delta y) - \tan y$$

$$= \frac{\sin(y + \delta y) . \cos y - \sin y . \cos(y + \delta y)}{\cos y . \cos(y + \delta y)}$$

$$= \frac{\sin \delta y}{\cos y . \cos(y + \delta y)},$$

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\sin \delta y} . \cos y . \cos(y + \delta y),$$

$$\frac{dy}{dx} = (\cos y)^2 = \frac{1}{1 + (\tan y)^2} = \frac{1}{1 + x^2},$$

or $$dy = \frac{dx}{1 + x^2}.$$

To find the Differential Coefficient of $\cot^{-1} x$.

42.
$$y = \cot^{-1} x, \quad y + \delta y = \cot^{-1}(x + \delta x),$$
$$\cot y = x, \quad \cot(y + \delta y) = x + \delta x,$$
$$\begin{aligned}\delta x &= \cot(y + \delta y) - \cot y \\ &= \frac{\sin y \cos(y + \delta y) - \cos y \sin(y + \delta y)}{\sin y \,.\, \sin(y + \delta y)} \\ &= \frac{-\sin \delta y}{\sin y \,.\, \sin(y + \delta y)},\end{aligned}$$
$$\frac{\delta y}{\delta x} = -\frac{\delta y}{\sin \delta y} \,.\, \sin y \,.\, \sin(y + \delta y),$$
$$\frac{dy}{dx} = -(\sin y)^2 = -\frac{1}{1 + (\cot y)^2} = -\frac{1}{1 + x^2},$$
$$dy = -\frac{dx}{1 + x^2}.$$

To find the Differential Coefficient of $\sec^{-1} x$.

43.
$$y = \sec^{-1} x, \qquad y + \delta y = \sec^{-1}(x + \delta x),$$
$$\sec y = x, \qquad \sec(y + \delta y) = x + \delta x,$$
$$\begin{aligned}\delta x &= \sec(y + \delta y) - \sec y \\ &= \frac{\cos y - \cos(y + \delta y)}{\cos y \,.\, \cos(y + \delta y)} \\ &= \frac{2 \sin \frac{\delta y}{2} \,.\, \sin\left(y + \frac{\delta y}{2}\right)}{\cos y \,.\, \cos(y + \delta y)},\end{aligned}$$
$$\frac{\delta y}{\delta x} = \frac{\frac{\delta y}{2}}{\sin \frac{\delta y}{2}} \,.\, \frac{\cos y \cos(y + \delta y)}{\sin\left(y + \frac{\delta y}{2}\right)},$$
$$\begin{aligned}\frac{dy}{dx} = \frac{(\cos y)^2}{\sin y} &= \frac{1}{(\sec y)^2} \,.\, \frac{1}{\sin y} \\ &= \frac{1}{(\sec y)^2} \,.\, \frac{\sec y}{\{(\sec y)^2 - 1\}^{\frac{1}{2}}} \\ &= \frac{1}{x(x^2 - 1)^{\frac{1}{2}}},\end{aligned}$$
$$dy = \frac{dx}{x(x^2 - 1)^{\frac{1}{2}}}.$$

To find the Differential Coefficient of $\operatorname{cosec}^{-1} x$.

44. $$y = \operatorname{cosec}^{-1} x, \qquad y + \delta y = \operatorname{cosec}^{-1}(x + \delta x),$$

$$\operatorname{cosec} y = x, \qquad \operatorname{cosec}(y + \delta y) = x + \delta x,$$

$$\delta x = \operatorname{cosec}(y + \delta y) - \operatorname{cosec} y$$

$$= \frac{\sin y - \sin(y + \delta y)}{\sin y \,.\, \sin(y + \delta y)}$$

$$= -\frac{2 \sin \frac{\delta y}{2} \,.\, \cos\left(y + \frac{\delta y}{2}\right)}{\sin y \,.\, \sin(y + \delta y)};$$

$$\frac{\delta y}{\delta x} = -\frac{\frac{\delta y}{2}}{\sin \frac{\delta y}{2}} \,.\, \frac{\sin y \sin(y + \delta y)}{\cos\left(y + \frac{\delta y}{2}\right)},$$

$$\frac{dy}{dx} = -\frac{(\sin y)^2}{\cos y} = -\frac{1}{(\operatorname{cosec} y)^2} \,.\, \frac{1}{\cos y}$$

$$= -\frac{1}{(\operatorname{cosec} y)^2} \,.\, \frac{\operatorname{cosec} y}{\{(\operatorname{cosec} y)^2 - 1\}^{\frac{1}{2}}}$$

$$= -\frac{1}{x(x^2 - 1)^{\frac{1}{2}}},$$

$$dy = -\frac{dx}{x(x^2 - 1)^{\frac{1}{2}}}.$$

Differentiation of Simple Functions of y *with regard to* x, *y being a function of x.*

45. Let $u = y^m$. Now, by Art. (19),

$$\frac{du}{dx} = \frac{du}{dy} \,.\, \frac{dy}{dx},$$

and, by Art. (30), $$\frac{du}{dy} = my^{m-1};$$

hence, if $u = y^m$, $$\frac{du}{dx} = my^{m-1}\frac{dy}{dx}.$$

Similarly it may be shewn that,

if $u = \log_a y$, $\qquad \dfrac{du}{dx} = \dfrac{1}{y \log_e a} \cdot \dfrac{dy}{dx}$:

if $u = a^y$, $\qquad \dfrac{du}{dx} = a^y \cdot \log_e a \cdot \dfrac{dy}{dx}$:

if $u = \sin y$, $\qquad \dfrac{du}{dx} = \cos y \cdot \dfrac{dy}{dx}$:

if $u = \cos y$, $\qquad \dfrac{du}{dx} = - \sin y \cdot \dfrac{dy}{dx}$:

if $u = \tan y$, $\qquad \dfrac{du}{dx} = (\sec y)^2 \cdot \dfrac{dy}{dx}$:

if $u = \cot y$, $\qquad \dfrac{du}{dx} = - (\operatorname{cosec} y)^2 \cdot \dfrac{dy}{dx}$:

if $u = \sec y$, $\qquad \dfrac{du}{dx} = \tan y \cdot \sec y \cdot \dfrac{dy}{dx}$:

if $u = \operatorname{cosec} y$, $\qquad \dfrac{du}{dx} = - \cot y \cdot \operatorname{cosec} y \cdot \dfrac{dy}{dx}$:

if $u = \sin^{-1} y$, $\qquad \dfrac{du}{dx} = \dfrac{1}{(1 - y^2)^{\frac{1}{2}}} \cdot \dfrac{dy}{dx}$:

if $u = \cos^{-1} y$, $\qquad \dfrac{du}{dx} = - \dfrac{1}{(1 - y^2)^{\frac{1}{2}}} \cdot \dfrac{dy}{dx}$:

if $u = \tan^{-1} y$, $\qquad \dfrac{du}{dx} = \dfrac{1}{1 + y^2} \cdot \dfrac{dy}{dx}$:

if $u = \cot^{-1} y$, $\qquad \dfrac{du}{dx} = - \dfrac{1}{1 + y^2} \cdot \dfrac{dy}{dx}$:

if $u = \sec^{-1} y$, $\qquad \dfrac{du}{dx} = \dfrac{1}{y (y^2 - 1)^{\frac{1}{2}}} \cdot \dfrac{dy}{dx}$:

if $u = \operatorname{cosec}^{-1} y$, $\qquad \dfrac{du}{dx} = - \dfrac{1}{y (y^2 - 1)^{\frac{1}{2}}} \cdot \dfrac{dy}{dx}$.

All these formulæ the student must carefully commit to memory.

SECTION III. ILLUSTRATIVE EXAMPLES.

45′. We shall devote this section to the exemplification of the principles which have been established in Sections (1) and (2). The illustrations here given are not numerous: in order to acquire a practical familiarity with the processes of differentiation, as well as with the application of the general theorems which we shall develop in the subsequent pages of this work, it will be necessary for the student to have recourse to Peacock's or Gregory's Examples of the Differential Calculus.

Ex. 1. Let $y = \sin \alpha + \sin \beta + \sin \gamma$, α, β, γ, not involving x; then, the sum of the three sines being a constant quantity, we have, by Art. (11),

$$\frac{dy}{dx} = 0.$$

Ex. 2. Let $$y = x^3 + a^3,$$

a being a constant quantity; then, by Arts. (12) and (30), we have

$$\frac{dy}{dx} = 3x^2.$$

Ex. 3. Let $$y = b \log_a x,$$

a and b being constants; then, by Arts. (13) and (31), we have

$$\frac{dy}{dx} = \frac{b}{x \log_e a}.$$

Ex. 4. Let $$y = x^a + a^x,$$

a being a constant; then, by Arts. (14), (30), and (32),

$$\frac{dy}{dx} = ax^{a-1} + a^x \log_e a.$$

Ex. 5. Let $$y = \sin x \,.\, \cos x;$$

then, by Arts. (15), (33), (34),

$$\frac{dy}{dx} = \sin x \frac{d \cos x}{dx} + \cos x \frac{d \sin x}{dx}$$
$$= \sin x (-\sin x) + \cos x \,.\, \cos x$$
$$= (\cos x)^2 - (\sin x)^2.$$

Ex. 6. Let $$y = \frac{\tan x}{x};$$

then, by Arts. (16) and (35), there is

$$\frac{dy}{dx} = \frac{1}{x^2}\left(x\frac{d\tan x}{dx} - \tan x\frac{dx}{dx}\right)$$

$$= \frac{1}{x^2}\left\{x(\sec x)^2 - \tan x\right\}.$$

Ex. 7. Let $y = x^n . \log_a x . a^x . \sin x . \sin^{-1} x$;

then, by Arts. (17), (30), (31), (32), (33), (39),

$$\frac{dy}{y} = \frac{d(x^n)}{x^n} + \frac{d\log_a x}{\log_a x} + \frac{d(a^x)}{a^x} + \frac{d\sin x}{\sin x} + \frac{d\sin^{-1} x}{\sin^{-1} x}$$

$$= \frac{n\,dx}{x} + \frac{dx}{x\log_a x\log_e a} + \log_e a\,dx + \frac{dx}{\tan x} + \frac{dx}{\sin^{-1} x(1-x^2)^{\frac{1}{2}}}.$$

Ex. 8. Let it be proposed to find $\dfrac{dx}{dy}$ in terms of x, having given that

$$y = \sin x.$$

By Art. (33), we have $$\frac{dy}{dx} = \cos x,$$

and, by Art. (18), $$\frac{dy}{dx}.\frac{dx}{dy} = 1:$$

hence $$\cos x . \frac{dx}{dy} = 1, \qquad \frac{dx}{dy} = \sec x.$$

Ex. 9. If $u = y^3$, and $y = \cot x$; let it be proposed to find $\dfrac{du}{dx}$.

We have, by Art. (30), $$\frac{du}{dy} = 3y^2,$$

and, by Art. (36), $$\frac{dy}{dx} = -(\operatorname{cosec} x)^2:$$

hence, by Art. (19),

$$\frac{du}{dx} = \frac{du}{dy}.\frac{dy}{dx} = -3y^2(\operatorname{cosec} x)^2 = -3(\cot x)^2(\operatorname{cosec} x)^2.$$

Ex. 10. Given

$$u = \sin z, \quad z = \sin y, \quad y = \sin x,$$

to find $\frac{du}{dx}$. We have, by Art. (33),

$$\frac{du}{dz} = \cos z, \ \frac{dz}{dy} = \cos y, \ \frac{dy}{dx} = \cos x:$$

but, by Art. (19), Cor.

$$\frac{du}{dx} = \frac{du}{dz}\frac{dz}{dy}\frac{dy}{dx};$$

hence
$$\frac{du}{dx} = \cos z \cos y \cos x.$$

Ex. 11. Given

$$u = \sin(ay) + \sin(\beta z) + \tan^{-1}(yz),$$
$$y = \sec x, \quad z = \operatorname{cosec} x,$$

to find $\frac{Du}{dx}$.

By Art. (20), $$\frac{Du}{dx} = \frac{du}{dy}\frac{dy}{dx} + \frac{du}{dz}\frac{dz}{dx}.$$

By Arts. (14) and (45),

$$\frac{du}{dy} = \cos(ay) \cdot \frac{d(ay)}{dy} + \frac{1}{1 + (yz)^2}\frac{d(yz)}{dy}$$
$$= a\cos(ay) + \frac{z}{1 + y^2z^2}:$$

similarly
$$\frac{du}{dz} = \beta\cos(\beta z) + \frac{y}{1 + y^2z^2}.$$

Also, by Arts. (37) and (38),

$$\frac{dy}{dx} = \tan x \sec x, \quad \frac{dz}{dx} = -\cot x \,.\, \operatorname{cosec} x.$$

Hence we have

$$\frac{Du}{dx} = \left\{a\cos(ay) + \frac{z}{1 + y^2z^2}\right\}\tan x \sec x$$
$$- \left\{\beta\cos(\beta z) + \frac{y}{1 + y^2z^2}\right\}\cot x \operatorname{cosec} x.$$

Ex. 12. Given that $u = y_1^{y_2^{y_3}}$,

and that $y_1 = x, \quad y_2 = x^2, \quad y_3 = x^3,$

to find $\dfrac{Du}{dx}$.

We have, by Art. (21),

$$\frac{Du}{dx} = \frac{du}{dy_1}\,\frac{dy_1}{dx} + \frac{du}{dy_2}\,\frac{dy_2}{dx} + \frac{du}{dy_3}\,\frac{dy_3}{dx}.$$

But $$\frac{du}{dy_1} = \frac{1}{y_1}\,.\,y_2^{y_3}.\,y_1^{y_2^{y_3}}, \text{ by Art. (30):}$$

$$\frac{du}{dy_2} = \log_e y_1\,.\,\frac{d\,(y_2^{y_3})}{dy_2}\,.\,y_1^{y_2^{y_3}}, \text{ by Art. (32),}$$

$$= \log_e y_1\,.\,y_3\,.\,\frac{y_2^{y_3}}{y_2}.\,y_1^{y_2^{y_3}}, \text{ by Art. (30);}$$

$$\frac{du}{dy_3} = \log_e y_1\,.\,y_1^{y_2^{y_3}}.\,\frac{d\,(y_2^{y_3})}{dy_3}, \text{ by Art. (32),}$$

$$= \log_e y_1\,.\,y_1^{y_2^{y_3}}.\,\log_e y_2\,.\,y_2^{y_3}, \text{ by Art. (32),}$$

$$= \log_e y_1\,.\,\log_e y_2\,.\,y_2^{y_3}.\,y_1^{y_2^{y_3}}.$$

Hence

$$\frac{Du}{dx} = \frac{1}{y_1}.y_2^{y_3}.y_1^{y_2^{y_3}} + 2x.\log_e y_1.y_3.\frac{y_2^{y_3}}{y_2}.y_1^{y_2^{y_3}} + 3x^2\log_e y_1.\log_e y_2.y_2^{y_3}.y_1^{y_2^{y_3}}$$

$$= y_2^{y_3}.\,y_1^{y_2^{y_3}}.\left(\frac{1}{x} + 2x^2\log_e y_1 + 3x^2\log_e y_1\,.\,\log_e y_2\right).$$

Ex. 13. Given that

$$u = (a^2 - x^2)^{\frac{1}{2}} + \sin y = 0,$$

to find $\dfrac{dy}{dx}$.

By Art. (22), $$\frac{du}{dx} + \frac{du}{dy}\,\frac{dy}{dx} = 0.$$

But, by Art. (45), when we differentiate considering y constant,

$$\frac{du}{dx} = \tfrac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}\,.\,\frac{d\,(a^2 - x^2)}{dx}$$

$$= \tfrac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}\,.\,(-\,2x) = -\,x\,(a^2 - x^2)^{-\frac{1}{2}}.$$

Also $\dfrac{du}{dy} = \cos y$.

Hence we have

$$-x\,(a^2 - x^2)^{-\frac{1}{2}} + \cos y\,.\,\frac{dy}{dx} = 0,$$

whence $$\frac{dy}{dx} = \frac{x}{\cos y\,(a^2 - x^2)^{\frac{1}{2}}}.$$

Ex. 14. Given that $$u = x^{yz},$$
z being a function of x and y by virtue of the equation
$$z = \sin (xy),$$
let it be proposed to find
$$\frac{du}{dx},\ \frac{du}{dy},\ \frac{du}{dz},\ \frac{Du}{dx},\ \frac{Du}{dy},\ Du.$$

By Art. (30), $\dfrac{du}{dx} = yz\,.\,\dfrac{x^{yz}}{x}$:

by Art. (45), $$\frac{du}{dy} = \log_e x\,.\,x^{yz}.\,\frac{d\,(yz)}{dy} = z\,.\,\log_e x\,.\,x^{yz},$$

and $$\frac{du}{dz} = \log_e x\,.\,x^{yz}.\,\frac{d\,(yz)}{dz} = y\,\log_e x\,.\,x^{yz}.$$

Also, by Art. (45),

$$\frac{dz}{dx} = \cos (xy)\,\frac{d\,(xy)}{dx} = y\cos (xy),$$

$$\frac{dz}{dy} = \cos (xy)\,\frac{d\,(xy)}{dy} = x\cos (xy).$$

But, by Art. (25),

$$\frac{Du}{dx} = \frac{du}{dx} + \frac{du}{dz}\,\frac{dz}{dx},$$

$$\frac{Du}{dy} = \frac{du}{dy} + \frac{du}{dz}\,\frac{dz}{dy}.$$

Hence we see that

$$\frac{Du}{dx} = \frac{yz}{x}\,.\,x^{yz} + y\,\log_e x\,.\,x^{yz}.\,y\cos (xy)$$

$$= yx^{yz}\left\{\frac{z}{x} + y\,\log_e x\,.\,\cos (xy)\right\};$$

$$\frac{Du}{dy} = z \log_e x \, . \, x^{yz} + y \log_e x \, . \, x^{yz} . \, x \cos (xy)$$

$$= \log_e x \, . \, x^{yz} . \, \{z + xy \cdot \cos (xy)\}.$$

Also

$$Du = \frac{Du}{dx} \, dx + \frac{Du}{dy} \, dy$$

$$= yx^{yz} \left\{\frac{z}{x} + y \log_e x \, . \cos (xy)\right\} dx + \log_e x \, . \, x^{yz} . \{z + xy \cos (xy)\} \, dy.$$

Ex. 15. Given that

$$u = \sin (xy + yz + zx) = 0,$$

to find $\frac{dz}{dx}$ and $\frac{dz}{dy}$.

By Art. (45),

$$\frac{du}{dx} = \cos (xy + yz + zx) \, . \, \frac{d \, (xy + yz + zx)}{dx}$$

$$= (y + z) \cos (xy + yz + zx):$$

similarly $$\frac{du}{dy} = (z + x) \cos (xy + yz + zx),$$

$$\frac{du}{dz} = (x + y) \cos (xy + yz + zx).$$

Hence, by the formulæ of Art. (27), viz.

$$\frac{du}{dx} + \frac{du}{dz} \, \frac{dz}{dx} = 0,$$

$$\frac{du}{dy} + \frac{du}{dz} \, \frac{dz}{dy} = 0,$$

we have $$y + z + (x + y) \frac{dz}{dx} = 0,$$

$$z + x + (x + y) \frac{dz}{dy} = 0 ;$$

whence $$\frac{dz}{dx} = -\frac{y+z}{x+y}, \quad \frac{dz}{dy} = -\frac{z+x}{x+y},$$

or, transforming partial differential coefficients into differentials,

$$d_x z = -\frac{y+z}{x+y}\,dx, \quad d_y z = -\frac{z+x}{x+y}\,dy.$$

CHAPTER III.

SUCCESSIVE DIFFERENTIATION.

Theory of the Independent Variable.

46. Let $\phi(x, y) = 0$, where $\phi(x, y)$ denotes any function of x and y whatever. When for x we substitute the successive values $x + \delta x$, $x + 2\delta x$, $x + 3\delta x$, ... let the corresponding values of y be $y_1, y_2, y_3, \ldots$. Then, δy denoting the increment of y due to the increment δx of x, we have

$$y_1 = y + \delta y:$$

hence, putting $y + \delta y$ for y and y_2 for y_1 in this equation, which corresponds to the change in the equation due to giving x another increment δx,

$$y_2 = y + \delta y + \delta(y + \delta y),$$

or, putting $\delta\delta y = \delta^2 y$, as an abbreviation of notation,

$$y_2 = y + 2\delta y + \delta^2 y.$$

Similarly, x receiving a third increment δx,

$$\begin{aligned} y_3 &= y + \delta y + 2\delta(y + \delta y) + \delta^2(y + \delta y) \\ &= y + 3\delta y + 3\delta^2 y + \delta^3 y. \end{aligned}$$

Proceeding in the same way, we shall finally get, the law of the coefficients being evidently the same as in the binomial theorem,

$$y_n = y + \frac{n}{1}\delta y + \frac{n(n-1)}{1.2}\delta^2 y + \ldots + \frac{n}{1}\delta^{n-1}y + \delta^n y.$$

Thus we see that as x keeps increasing by equal increments δx, y generally increases by unequal increments: in fact the increment of y, corresponding to an increment δx of x, is δy, and, for an increment $n\delta x$ of x, it is not $n\delta y$, but

$$\frac{n}{1}\delta y + \frac{n(n-1)}{1.2}\delta^2 y + \ldots + \frac{n}{1}\delta^{n-1}y + \delta^n y.$$

The quantity x, which is supposed to increase by equal augments, is called the *independent* variable, while y, the increments of which are dependent upon those of x, and which are generally variable, is called the *dependent* variable. Such is the definition of an independent and a dependent variable in the calculus of finite differences. Suppose now the difference δx to be indefinitely diminished, then we may replace

$$\delta x,\ \delta y,\ \delta^2 y,\ \delta^3 y, \ldots\ldots$$

by the differentials

$$dx,\ dy,\ d^2 y,\ d^3 y, \ldots\ldots$$

which are proportional to them. We may then say, to adapt our definitions to the differential calculus, that if y be a function of x, x will be the independent and y the dependent variable, if, while x varies, its differential dx remains constant: in accordance with this definition not only y but also dy will generally vary with the variation of x.

Ex. 1. Let $y = \sin x$; then, x being the independent variable,

$$dy = \cos x \,.\, dx:$$

differentiating again, x and dy being variable, and dx constant,

$$\begin{aligned} d^2 y &= d(\cos x) \,.\, dx \\ &= (-\sin x \,.\, dx) \,.\, dx \\ &= -\sin x \,.\, dx^2, \end{aligned}$$

where dx^2, for simplicity of writing, is put instead of $(dx)^2$. Proceeding in the same way, we see that

$$d^n y = (-)^{\frac{n}{2}} . \sin x \,.\, dx^n,\ n \text{ being even};$$

$$d^n y = (-)^{\frac{n-1}{2}} . \cos x \,.\, dx^n,\ n \text{ being odd}.$$

These expressions may be written also thus:

$$\frac{d^n y}{dx^n} = (-)^{\frac{n}{2}} . \sin x,\ n \text{ being even}:$$

$$\frac{d^n y}{dx^n} = (-)^{\frac{n-1}{2}} . \cos x,\ n \text{ being odd}.$$

Ex. 2. Suppose that

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots + a_n x^n,$$

a rational function of x of n dimensions: then, differentiating successively n times, we have, x being the independent variable,

$$\frac{dy}{dx} = 1.a_1 + 2.a_2.x + 3.a_3.x^2 + \ldots\ldots + na_n x^{n-1},$$

$$\frac{d^2y}{dx^2} = 1.2.a_2 + 2.3.a_3.x + \ldots\ldots + (n-1)\,na_n x^{n-2},$$

$$\frac{d^3y}{dx^3} = 1.2.3.a_3 + \ldots\ldots + (n-2)(n-1)\,na_n x^{n-3},$$

. . . .

. . . .

$$\frac{d^ny}{dx^n} = 1.2.3.\ldots\ldots n.a_n.$$

The differential coefficients of higher orders than the nth, viz.

$$\frac{d^{n+1}y}{dx^{n+1}},\ \frac{d^{n+2}y}{dx^{n+2}},\ \frac{d^{n+3}y}{dx^{n+3}},\ldots.$$

will all be zero.

Ex. 3. To find the nth differential coefficient of $\frac{1}{x^2-a^2}$.

Put $$y = \frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$= \frac{1}{2a}\left\{(x-a)^{-1} - (x+a)^{-1}\right\}.$$

Then

$$\frac{dy}{dx} = \frac{1}{2a}(-1)\left\{(x-a)^{-2} - (x+a)^{-2}\right\},$$

$$\frac{d^2y}{dx^2} = \frac{1}{2a}(-1)(-2)\left\{(x-a)^{-3} - (x+a)^{-3}\right\},$$

. . . .

. . . .

$$\frac{d^ny}{dx^n} = \frac{(-1)^n}{2a}.1.2.3.\ldots\ldots n.\left\{(x-a)^{-n-1} - (x+a)^{-n-1}\right\}.$$

Change of the Independent Variable.

47. Suppose that we have an equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots\right) = 0,$$

involving x, y, and successive differentials of y taken on the hypothesis that dx is constant. It is frequently desirable in researches in the differential calculus to transform this differential equation into an equivalent one in which, instead of x, some quantity θ of which x is a function, shall be the independent variable. On the new hypothesis dx will no longer generally be constant.

Suppose that $y = f(x)$, and put

$$\frac{df(x)}{dx} = f'(x),$$

$f'(x)$ being another function of x: adopting the same notation, put

$$\frac{df'(x)}{dx} = f''(x),$$

and so on. The quantities $f'(x)$, $f''(x)$, $f'''(x)$,......are called the first, second, third,....*derived functions* or *derivatives* of $f(x)$, and are certain algebraical expressions constituting the results of the operations upon the function $f(x)$ designated by the differential coefficients

$$\frac{df(x)}{dx},\ \frac{d^2f(x)}{dx^2},\ \frac{d^3f(x)}{dx^3},\ldots.$$

Then, taking dy, d^2y, d^3y,.... to represent the differentials of y on the hypothesis that dx is constant, and $d'y$, d'^2y, d'^3y,.... its differentials, supposing dx to vary, we have

$$\left.\begin{aligned} dy &= f'(x).dx \\ d'y &= f'(x).dx \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1):$$

$$\left.\begin{aligned} d^2y &= f''(x)\,dx^2 \\ d'^2y &= f''(x)\,dx^2 + f'(x)\,d^2x \end{aligned}\right\}\ldots\ldots\ldots\ldots(2):$$

$$\left.\begin{aligned} d^3y &= f'''(x)\,dx^3 \\ d'^3y &= f'''(x)\,dx^3 + 3f''(x)\,dxd^2x + f'(x)\,d^3x \end{aligned}\right\}\ldots(3):$$

and so on to any order of differentiation.

Now, by the aid of the relation subsisting between x and θ, dx, d^2x, d^3x,....may be found in terms of θ and $d\theta$, and therefore, from (1), (2), (3),.... we can obtain dy, d^2y, d^3y,.... in terms of $d'y$, d'^2y, d'^3y.... , θ, $d\theta$.

From (1) we see that $dy = d'y \dots\dots\dots\dots\dots\dots (4):$

from (1) and (2) $$d'^2y = d^2y + \frac{d'y}{dx}\,d^2x,$$

$$d^2y = \frac{dx\,d'^2y - d^2x\,d'y}{dx} \dots\dots\dots\dots (5):$$

from (1), (2), (3), we may get also

$$d^3y = \frac{dx\,(dxd'^3y - d^3xd'y) - 3d^2x\,(dxd'^2y - d^2xd'y)}{dx^2} \dots (6):$$

and so on for d^4y, d^5y,......

The second equations of the systems (1), (2), (3),......may be written in a form which may serve to suggest to the memory that θ is the independent variable corresponding to the differentials $d'y$, d'^2y, d'^3y,...... of y, viz.

$$\frac{d'y}{d\theta} = f'(x)\,\frac{dx}{d\theta} \dots\dots\dots\dots\dots\dots\dots\dots\dots (7),$$

$$\frac{d'^2y}{d\theta^2} = f''(x)\,\frac{dx^2}{d\theta^2} + f'(x)\,\frac{d^2x}{d\theta^2} \dots\dots\dots\dots\dots\dots (8),$$

$$\frac{d'^3y}{d\theta^3} = f'''(x)\,\frac{dx^3}{d\theta^3} + 3f''(x)\,\frac{dx}{d\theta}\,\frac{d^2x}{d\theta^2} + f'(x)\,\frac{d^3x}{d\theta^3} \dots\dots (9),$$

. . . .

If we substitute in the differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \dots\right) = 0,$$

the values of $$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots$$

that is, the values of $f'(x)$, $f''(x)$, $f'''(x)$,....

obtained from the equations (7), (8), (9),.... we shall have transformed the equation into an equivalent one

$$\Psi\left(x, y, \frac{dx}{d\theta}, \frac{d^2x}{d\theta^2}, \dots, \frac{d'y}{d\theta}, \frac{d'^2y}{d\theta^2}, \dots\right) = 0,$$

or, eliminating $x, \dfrac{dx}{d\theta}, \dfrac{d^2x}{d\theta^2}, \ldots\ldots$ by the aid of the relation between x and θ,

$$\chi\left(\theta, y, \frac{d'y}{d\theta}, \frac{d'^2y}{d\theta^2}, \ldots\right) = 0.$$

COR. Suppose that our object is to change the independent variable from x to y: then, by the formulæ (4), (5), (6), considering $d'y$ constant, and therefore equating $d'^2y, d'^3y, \ldots$ to zero, we have

$$dy = d'y, \quad d^2y = -\frac{d'y\,d^2x}{dx},$$

$$d^3y = \frac{-dx\,d^3x\,d'y + 3(d^2x)^2d'y}{dx^2},$$

and so on.

Ex. 1. Given $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + n^2y = 0$:

to change the independent variable from x to θ, when $x = \cos\theta$.

In this case $d'y = f'(x)\,dx = -f'(x)\sin\theta\,d\theta$;

differentiating again, considering $d\theta$ constant, we have

$$d'^2y = -f''(x)\,dx\sin\theta\,d\theta - f'(x)\cos\theta\,d\theta^2$$
$$= f''(x)\sin^2\theta\,d\theta^2 - f'(x)\cos\theta\,d\theta^2,$$

or

$$\frac{d'^2y}{d\theta^2} = (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx},$$

and therefore the transformed equation is

$$\frac{d'^2y}{d\theta^2} + n^2y = 0.$$

Ex. 2. To change the independent variable in

$$(a+x)^3\frac{d^3y}{dx^3} + 3(a+x)^2\frac{d^2y}{dx^2} + (a+x)\frac{dy}{dx} + by = 0,$$

from x to θ, having given

$$\theta = \log(a+x).$$

In this case

$$\begin{aligned} d'y &= f'(x)\,dx = f'(x)\,e^{\theta}\,d\theta, \\ d'^2y &= f''(x)\,dx\,e^{\theta}\,d\theta + f'(x)\,e^{\theta}\,d\theta^2 \\ &= f''(x)\,e^{2\theta}\,d\theta^2 + f'(x)\,e^{\theta}\,d\theta^2, \\ d'^3y &= f'''(x)\,dx\,e^{2\theta}\,d\theta^2 + 2f''(x)\,e^{2\theta}\,d\theta^3 + f''(x)\,dx\,e^{\theta}\,d\theta^2 + f'(x)\,e^{\theta}\,d\theta^3 \\ &= f'''(x)\,e^{3\theta}\,d\theta^3 + 3f''(x)\,e^{2\theta}\,d\theta^3 + f'(x)\,e^{\theta}\,d\theta^3, \end{aligned}$$

or $$\frac{d'^3y}{d\theta^3} = (a+x)^3\frac{d^3y}{dx^3} + 3(a+x)^2\frac{d^2y}{dx^2} + (a+x)\frac{dy}{dx},$$

which reduces the proposed equation to the form

$$\frac{d'^3y}{d\theta^3} + by = 0.$$

Order of Partial Differentiations indifferent.

48. The following is a theorem of great importance in successive differentiation: if

$$u = f(y_1, y_2),$$

then $$d_{y_2}d_{y_1}u = d_{y_1}d_{y_2}u.$$

For $$\delta_{y_1}u = f(y_1 + \delta y_1, y_2) - f(y_1, y_2),$$

and therefore

$$\begin{aligned} \delta_{y_2}\delta_{y_1}u &= \{f(y_1 + \delta y_1, y_2 + \delta y_2) - f(y_1, y_2 + \delta y_2)\} \\ &\quad - \{f(y_1 + \delta y_1, y_2) - f(y_1, y_2)\} \\ &= f(y_1+\delta y_1, y_2+\delta y_2) - f(y_1, y_2+\delta y_2) - f(y_1+\delta y_1, y_2) + f(y_1, y_2). \end{aligned}$$

In precisely the same way it may be shewn that

$$\delta_{y_1}\delta_{y_2}u = f(y_1+\delta y_1, y_2+\delta y_2) - f(y_1+\delta y_1, y_2) - f(y_1, y_2+\delta y_2) + f(y_1, y_2).$$

The right-hand members of these last two equations being identical, we must have also

$$\delta_{y_2}\delta_{y_1}u = \delta_{y_1}\delta_{y_2}u.$$

This relation is true whatever be the magnitudes of δy_1 and δy_2: if we proceed to the limit, by taking δy_1 and δy_2 less than any assignable magnitudes, and replace infinitesimal differences by differentials, we get

$$d_{y_2}d_{y_1}u = d_{y_1}d_{y_2}u.$$

Expressing the theorem by partial differential coefficients instead of partial differentials, we have

$$\frac{d_{y_2} d_{y_1} u}{dy_2 dy_1} = \frac{d_{y_1} d_{y_2} u}{dy_1 dy_2};$$

or, as these partial differential coefficients are ordinarily expressed for the sake of brevity,

$$\frac{d^2 u}{dy_2 dy_1} = \frac{d^2 u}{dy_1 dy_2}.$$

Cor. 1. By virtue of the theorem

$$d_{y_2} d_{y_1} u = d_{y_1} d_{y_2} u,$$

it is evident that the symbols d_{y_1}, d_{y_2}, of partial differentiation, may be permuted in every possible way: thus

$$d^2{}_{y_1} d_{y_2} u = d_{y_1} d_{y_1} d_{y_2} u = d_{y_1} d_{y_2} d_{y_1} u = d_{y_2} d_{y_1} d_{y_1} u = d_{y_2} d^2{}_{y_1} u,$$

or, in the language of partial differential coefficients,

$$\frac{d^3 u}{dy_1^2 dy_2} = \frac{d^3 u}{dy_1 dy_2 dy_1} = \frac{d^3 u}{dy_2 dy_1^2}.$$

Cor. 2. The theorem which we have established in relation to partial differentiation of functions of two variables, may evidently be extended to the general case of a function of any number of variables: thus, in an expression

$$d_{y_1}^{m_1} d_{y_2}^{m_2} d_{y_3}^{m_3} \ldots\ldots u,$$

u being a function of y_1, y_2, y_3, ... the symbols d_{y_1}, d_{y_2}, d_{y_3},. ... may be permuted *inter se* in the same way as the symbols of quantity, A_1, A_2, A_3,. ... in an algebraical product

$$A_1^{m_1}.\, A_2^{m_2}.\, A_3^{m_3}.\ldots$$

Ex. Let $u = y_1^{y_2}$: then

$$\frac{du}{dy_1} = y_2 \cdot y_1^{y_2-1},$$

$$\frac{d^2 u}{dy_2 dy_1} = y_1^{y_2-1} + y_2 \cdot \log(y_1) \cdot y_1^{y_2-1}. \frac{d(y_2-1)}{dy_2}$$

$$= y_1^{y_2-1} + y_2 \cdot \log(y_1) \cdot y_1^{y_2-1}.$$

Again
$$\frac{du}{dy_2} = \log y_1 \cdot y_1^{y_2},$$

$$\frac{d^2u}{dy_1\,dy_2} = \frac{1}{y_1} \cdot y_1^{y_2} + \log(y_1) \cdot y_2 \cdot y_1^{y_2-1}$$

$$= y_1^{y_2-1} + y_2 \cdot \log(y_1) \cdot y_1^{y_2-1}.$$

Thus we see that the results are the same for both orders of differentiation.

Ex. 2. Let
$$u = \sin(xy).$$

Then
$$\frac{du}{dx} = y\cos(xy),$$

$$\frac{d^2u}{dx^2} = -y^2\sin(xy),$$

$$\frac{d^3u}{dydx^2} = -2y\sin(xy) - xy^2\cos(xy).$$

Also
$$\frac{du}{dx} = y\cos(xy),$$

$$\frac{d^2u}{dydx} = \cos(xy) - xy\sin(xy),$$

$$\frac{d^3u}{dxdydx} = -y\sin(xy) - y\sin(xy) - xy^2\cos(xy)$$

$$= -2y\sin(xy) - xy^2\cos(xy).$$

Thus we see that
$$\frac{d^3u}{dydx^2} = \frac{d^3u}{dxdydx},$$

or, in the language of differentials,
$$d_y d_x^2 u = d_x d_y d_x u.$$

Successive Differentiation of an explicit Function of two Functions of a single Variable.

49. Let $u = f(y_1, y_2)$, y_1 and y_2 being each of them a function of x: then, by Art. (20),

$$\frac{Du}{dx} = \frac{du}{dy_1} \cdot \frac{dy_1}{dx} + \frac{du}{dy_2} \cdot \frac{dy_2}{dx} \quad \ldots\ldots\ldots\ldots (1).$$

Differentiating again, x being considered the independent variable, and observing that, for convenience of writing, we may put, V being any expression functional of x,

$$\frac{DV}{dx} = \frac{D}{dx} . V,$$

we have

$$\frac{D^2u}{dx^2} = \frac{D}{dx}\left(\frac{du}{dy_1} . \frac{dy_1}{dx}\right) + \frac{D}{dx}\left(\frac{du}{dy_2} . \frac{dy_2}{dx}\right)$$

$$= \frac{D}{dx}\left(\frac{du}{dy_1}\right) . \frac{dy_1}{dx} + \frac{du}{dy_1} . \frac{D}{dx}\left(\frac{dy_1}{dx}\right)$$

$$+ \frac{D}{dx}\left(\frac{du}{dy_2}\right) . \frac{dy_2}{dx_2} + \frac{du}{dy_2} . \frac{D}{dx}\left(\frac{dy_2}{dx}\right).$$

But, since y_1 and y_2 are functions of x only, and not of any functions of x, it follows that

$$\frac{D}{dx} . \frac{dy_1}{dx} = \frac{d}{dx} . \frac{dy_1}{dx} = \frac{d^2y_1}{dx^2}, \quad \text{and} \quad \frac{D}{dx} . \frac{dy_2}{dx} = \frac{d}{dx} . \frac{dy_2}{dx} = \frac{d^2y_2}{dx^2}:$$

hence

$$\frac{D^2u}{dx^2} = \frac{D}{dx}\left(\frac{du}{dy_1}\right) . \frac{dy_1}{dx} + \frac{D}{dx}\left(\frac{du}{dy_2}\right) . \frac{dy_2}{dx} + \frac{du}{dy_1} . \frac{d^2y_1}{dx^2} + \frac{du}{dy_2} . \frac{d^2y_2}{dx^2}.$$

Now $\frac{du}{dy_1}$ is equivalent to a function of y_1 and y_2 only, not involving the differential dy_1: thus, for instance, if $u = y_1^2y_2^2$, then $\frac{du}{dy_1} = 2y_1y_2^2$, where dy_1 does not appear. It follows therefore that in the expression

$$\frac{D}{dx}\left(\frac{du}{dy_1}\right)$$

we may regard dy_1 constant without affecting results. Hence, $\frac{du}{dy_1}$ now occupying the place of u in (1), this formula gives

$$\frac{D}{dx}\left(\frac{du}{dy_1}\right) = \frac{d}{dy_1}\left(\frac{du}{dy_1}\right) . \frac{dy_1}{dx} + \frac{d}{dy_2}\left(\frac{du}{dy_1}\right) . \frac{dy_2}{dx}$$

$$= \frac{d^2u}{dy_1^2} . \frac{dy_1}{dx} + \frac{d^2u}{dy_2dy_1} . \frac{dy_2}{dx}$$

$$= \frac{d^2u}{dy_1^2} . \frac{dy_1}{dx} + \frac{d^2u}{dy_1dy_2} . \frac{dy_2}{dx}.$$

Similarly, we must have

$$\frac{D}{dx}\left(\frac{du}{dy_2}\right)=\frac{d^2u}{dy_2^{\,2}}\cdot\frac{dy_2}{dx}+\frac{d^2u}{dy_1dy_2}\cdot\frac{dy_1}{dx}.$$

Hence we obtain

$$\frac{D^2u}{dx^2}=\frac{d^2u}{dy_1^{\,2}}\cdot\frac{dy_1^{\,2}}{dx^2}+2\,\frac{d^2u}{dy_1dy_2}\cdot\frac{dy_1}{dx}\cdot\frac{dy_2}{dx}+\frac{d^2u}{dy_2^{\,2}}\cdot\frac{dy_2^{\,2}}{dx^2}$$
$$+\frac{du}{dy_1}\cdot\frac{d^2y_1}{dx^2}+\frac{du}{dy_2}\cdot\frac{d^2y_2}{dx^2},$$

or

$$D^2u=\frac{d^2u}{dy_1^{\,2}}\cdot dy_1^{\,2}+2\,\frac{d^2u}{dy_1dy_2}\cdot dy_1.dy_2+\frac{d^2u}{dy_2^{\,2}}.dy_2^{\,2}+\frac{du}{dy_1}\cdot d^2y_1+\frac{du}{dy_2}.d^2y_2.$$

We might proceed in the same way to find the expressions for D^3u, D^4u,......; the formulæ however rapidly rise into tedious polynomials. We have confined our attention to the successive differentiation of a function of two functions; the extension however of the theory to a function of any number of functions is too obvious to present any difficulty to the student. Thus, supposing that

$$u=f(y_1, y_2, y_3),$$

the student will easily find that

$$\frac{D^2u}{dx^2}=\frac{d^2u}{dy_1^{\,2}}\cdot\frac{dy_1^{\,2}}{dx^2}+\frac{d^2u}{dy_2^{\,2}}\cdot\frac{dy_2^{\,2}}{dx^2}+\frac{d^2u}{dy_3^{\,2}}\cdot\frac{dy_3^{\,2}}{dx^2}$$
$$+2\,\frac{d^2u}{dy_2dy_3}\cdot\frac{dy_2}{dx}\cdot\frac{dy_3}{dx}+2\,\frac{d^2u}{dy_3dy_1}\cdot\frac{dy_3}{dx}\cdot\frac{dy_1}{dx}+2\,\frac{d^2u}{dy_1dy_2}\,\frac{dy_1}{dx}\,\frac{dy_2}{dx}$$
$$+\frac{du}{dy_1}\cdot\frac{d^2y_1}{dx^2}+\frac{du}{dy_2}\cdot\frac{d^2y_2}{dx^2}+\frac{du}{dy_3}\cdot\frac{d^2y_3}{dx^2}.$$

Ex. Let $u=\sin(y_1+y_2)$, $y_1=x$, $y_2=x^2$.

Then

$$\frac{du}{dy_1}=\cos(y_1+y_2),\quad \frac{du}{dy_2}=\cos(y_1+y_2),$$

$$\frac{d^2u}{dy_1^{\,2}}=-\sin(y_1+y_2),\ \frac{d^2u}{dy_1dy_2}=-\sin(y_1+y_2),\ \frac{d^2u}{dy_2^{\,2}}=-\sin(y_1+y_2):$$

hence

$$D^2u = -\sin(y_1+y_2)(dy_1^2+2\,dy_1\,dy_2+dy_2^2)+\cos(y_1+y_2).(d^2y_1+d^2y_2).$$

But

$$dy_1 = dx, \qquad d^2y_1 = 0,$$
$$dy_2 = 2x\,dx, \qquad d^2y_2 = 2dx^2:$$

hence

$$D^2u = -\sin(x+x^2).(dx^2+4x\,dx^2+4x^2\,dx^2)+2\cos(x+x^2)\,dx^2,$$

$$\frac{D^2u}{dx^2} = -\sin(x+x^2).(1+4x+4x^2)+2\cos(x+x^2).$$

We might of course have obtained this result by first giving y_1 and y_2 their values in the expression for u and then differentiating: thus

$$u = \sin(x+x^2),$$

$$\frac{Du}{dx} = \cos(x+x^2).(1+2x),$$

$$\frac{D^2u}{dx^2} = -\sin(x+x^2).(1+2x)^2+2\cos(x+x^2).$$

We may remark that, when u is expressed entirely in x, as in the latter method of differentiating u, the expressions

$$\frac{du}{dx}, \quad \frac{d^2u}{dx^2},$$

are equivalent to

$$\frac{Du}{dx}, \quad \frac{D^2u}{dx^2};$$

whereas, when we put

$$u = \sin(y_1+y_2),$$

$\frac{du}{dx}$, and $\frac{d^2u}{dx^2}$, are both zero, being in fact the first and second partial differential coefficients of u with respect to x, a letter not appearing in $\sin(y_1+y_2)$.

Successive Differentiation of an implicit Function of a single Variable.

50. Suppose that $v = \phi(x, y) = 0$,

y being thus an implicit function of x.

By Art. (22), we have

$$\frac{Dv}{dx} = \frac{dv}{dx} + \frac{dv}{dy} \cdot \frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating again, x being considered the independent variable, we have, $\frac{Dv}{dx}$ now taking the place of v,

$$\frac{D^2v}{dx^2} = \frac{D}{dx}\left(\frac{dv}{dx}\right) + \frac{D}{dx}\left(\frac{dv}{dy}\right) \cdot \frac{dy}{dx} + \frac{dv}{dy}\,\frac{D}{dx}\left(\frac{dy}{dx}\right) = 0.$$

But, by Art. (21), Cor.

$$\frac{D}{dx}\left(\frac{dv}{dx}\right) = \frac{d}{dx}\left(\frac{dv}{dx}\right) + \frac{d}{dy}\left(\frac{dv}{dx}\right)\frac{dy}{dx}$$

$$= \frac{d^2v}{dx^2} + \frac{d^2v}{dydx} \cdot \frac{dy}{dx}$$

$$= \frac{d^2v}{dx^2} + \frac{d^2v}{dxdy} \cdot \frac{dy}{dx},$$

and $$\frac{D}{dx}\left(\frac{dv}{dy}\right) = \frac{d}{dx}\left(\frac{dv}{dy}\right) + \frac{d}{dy}\left(\frac{dv}{dy}\right)\frac{dy}{dx}$$

$$= \frac{d^2v}{dxdy} + \frac{d^2v}{dy^2}\,\frac{dy}{dx}.$$

Also $$\frac{D}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

Hence $$\frac{D^2v}{dx^2} = \frac{d^2v}{dx^2} + 2\,\frac{d^2v}{dxdy} \cdot \frac{dy}{dx} + \frac{d^2v}{dy^2}\,\frac{dy^2}{dx^2} + \frac{dv}{dy}\,\frac{d^2y}{dx^2} \ldots\ldots (2),$$

or $$D^2v = \frac{d^2v}{dx^2}\,dx^2 + 2\,\frac{d^2v}{dxdy} \cdot dxdy + \frac{d^2v}{dy^2}\,dy^2 + \frac{dv}{dy} \cdot d^2y.$$

From (1) we may get $\frac{dy}{dx}$ in terms of the first order of partial differential coefficients of v, and therefore, from (2), we may get $\frac{d^2y}{dx^2}$ in terms of the first and second order of these coefficients. The partial differential coefficients of v may be obtained in terms of x and y: hence $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, may be found in terms of these two letters. We might proceed in the same way, by successive differentiation, to determine the third, fourth, fifth, &c. differential coefficients of y. The principles of this article the student will have no difficulty in extending to the successive differentiation of the system of equations considered in Art. (23).

Successive Total Differentials.

51. Let $u = f(y_1, y_2)$, y_1 and y_2 being independent variables: then, by Art. (24),

$$Du = d_{y_1}u + d_{y_2}u \ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating again we have

$$D^2u = Dd_{y_1}u + Dd_{y_2}u \ldots\ldots\ldots\ldots\ldots (2).$$

But, $d_{y_1}u$ being a function of y_1, y_2, and a constant dy_1, we have, by virtue of (1), $d_{y_1}u$ occupying the place of u,

$$Dd_{y_1}u = d_{y_1}d_{y_1}u + d_{y_2}d_{y_1}u$$

$$= d^2{}_{y_1}u + d_{y_1}d_{y_2}u.$$

Similarly $\qquad Dd_{y_2}u = d^2{}_{y_2}u + d_{y_1}d_{y_2}u.$

Hence, from (2), $\qquad D^2u = d^2{}_{y_1}u + 2d_{y_1}d_{y_2}\,u + d^2{}_{y_2}u.$

Proceeding in the same way, we easily see that

$$D^3u = d^3{}_{y_1}u + 3\,d^2{}_{y_1}d_{y_2}u + 3\,d_{y_1}d^2{}_{y_2}u + d^3{}_{y_3}u,$$

and so on, the law of the symbols of differentiation corresponding to the development of the binomial theorem: thus

$$D^nu = d^n{}_{y_1}u + \frac{n}{1}\,d_{y_1}{}^{n-1}d_{y_2}u + \frac{n(n-1)}{1.2}\,d_{y_1}{}^{n-2}\,d^2{}_{y_2}u + \ldots + nd_{y_1}d_{y_2}{}^{n-1}u + d_{y_2}{}^nu.$$

This relation may be expressed symbolically, thus:

$$D^nu = (d_{y_1} + d_{y_2})^n\,u.$$

Cor. If $u = f(y_1, y_2, y_3, \ldots y_m)$, it may be proved in a similar way, viz. by induction, that

$$D^nu = (d_{y_1} + d_{y_2} + d_{y_3} + \ldots\ldots + d_{y_m})^n\,u.$$

We may however establish this proposition by the following reasoning. By Art. (24),

$$Du = (d_{y_1} + d_{y_2} + d_{y_3} + \ldots\ldots + d_{y_m})\,u\,;$$

which shews that the symbol of operation D is equal to the sum of the operative symbols $d_{y1}, d_{y_2}, d_{y_3}, \ldots . d_{ym}$; hence

$$D^2u = D(d_{y_1} + d_{y2} + d_{y_3} + \ldots . + d_m)\, u$$
$$= (d_{y_1} + d_{y_2} + d_{y_3} + \ldots d_{y_m})(d_{y_1} + d_{y_2} + d_{y_3} + \ldots + d_{y_m})\, u \ldots (3).$$

But the symbols $d_{y_1}, d_{y_2}, d_{y_3}, \ldots . d_{y_m}$, are subject *inter se* to all the laws of combination which belong to symbols of quantity: thus for instance

$$d_{y_1}d_{y_2} = d_{y_2}d_{y_1}, \qquad d_{y_1}d_{y_1} = d^2_{y_1}, \qquad d_{y_1} + d_{y_2} = d_{y_2} + d_{y_1};$$

hence the product of the operative polynomials in (3) is equivalent to

$$(d_{y_1} + d_{y_2} + d_{y_3} + \ldots . + d_{y_m})^2:$$

proceeding successively to the higher orders to total differentials, we get the general formula

$$D^n u = (d_{y_1} + d_{y_2} + d_{y_3} + \ldots . + d_{y_m})^n\, u.$$

Successive Differentiation of an explicit Function of three Variables one of which is a Function of the other two.

52. Suppose that $u = f(x, y, z)$,

z being a function of x and y, two independent variables. Then, by Art. (25),

$$\frac{Du}{dx} = \frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\frac{Du}{dy} = \frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Differentiating (1) with regard to x, we have

$$\frac{D^2u}{dx^2} = \frac{D}{dx}\left(\frac{du}{dx}\right) + \frac{D}{dx}\left(\frac{du}{dz}\right) \cdot \frac{dz}{dx} + \frac{du}{dz}\,\frac{D}{dx}\left(\frac{dz}{dx}\right) \ldots\ldots (3).$$

But, $\frac{du}{dx}$ being a function of x, y, z, we have, by virtue of (1), putting $\frac{du}{dx}$ in place of u,

$$\frac{D}{dx}\left(\frac{du}{dx}\right) = \frac{d^2u}{dx^2} + \frac{d^2u}{dzdx} \cdot \frac{dz}{dx}:$$

similarly
$$\frac{D}{dx}\left(\frac{du}{dz}\right) = \frac{d^2u}{dxdz} + \frac{d^2u}{dz^2}\,\frac{dz}{dx}.$$

Also, since z is a function of x and y alone, and since the expression

$$\frac{D}{dx}\left(\frac{dz}{dx}\right),$$

denotes the total differential coefficient of $\frac{dz}{dx}$, a function of x and y, only so far as the variation of $\frac{dz}{dx}$ is affected by the variation of x when y remains constant, it is plain that

$$\frac{D}{dx}\left(\frac{dz}{dx}\right)=\frac{d^2z}{dx^2},$$

where $\frac{d^2z}{dx^2}$ is the second partial differential coefficient of z with regard to x.

Hence, from (3),

$$\frac{D^2u}{dx^2}=\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdz}\frac{dz}{dx}+\frac{d^2u}{dz^2}\frac{dz^2}{dx^2}+\frac{du}{dz}\frac{d^2z}{dx^2}.$$

In like manner, from (2) there is

$$\frac{D^2u}{dy^2}=\frac{d^2u}{dy^2}+2\frac{d^2u}{dydz}\frac{dz}{dy}+\frac{d^2u}{dz^2}\frac{dz^2}{dy^2}+\frac{du}{dz}\frac{d^2z}{dy^2}.$$

Again, from (2),

$$\frac{D^2u}{dxdy}=\frac{D}{dx}\left(\frac{du}{dy}\right)+\frac{D}{dx}\left(\frac{du}{dz}\right)\cdot\frac{dz}{dy}+\frac{du}{dz}\cdot\frac{D}{dx}\left(\frac{dz}{dy}\right).$$

But, from (1), putting $\frac{du}{dy}$, $\frac{du}{dz}$, successively for u,

$$\frac{D}{dx}\left(\frac{du}{dy}\right)=\frac{d^2u}{dxdy}+\frac{d^2u}{dzdy}\cdot\frac{dz}{dx},$$

$$\frac{D}{dx}\left(\frac{du}{dz}\right)=\frac{d^2u}{dxdz}+\frac{d^2u}{dz^2}\cdot\frac{dz}{dx}:$$

hence

$$\frac{D^2u}{dxdy}=\frac{d^2u}{dxdy}+\frac{d^2u}{dydz}\cdot\frac{dz}{dx}+\frac{d^2u}{dxdz}\cdot\frac{dz}{dy}+\frac{d^2u}{dz^2}\cdot\frac{dz}{dx}\cdot\frac{dz}{dy}+\frac{du}{dz}\cdot\frac{d^2z}{dxdy}.$$

We have therefore determined formulæ for the values of

$$\frac{D^2u}{dx^2},\quad\frac{D^2u}{dxdy},\quad\frac{D^2u}{dy^2}:$$

we might proceed to find the differential coefficients of higher orders by a continuation of precisely the same kind of processes.

COR. Suppose that $u = f(x, y, z) = 0$,

there being no other equation connecting x and y: z will thus be a function of x and y. Then, from the equations of Art. (27),

$$\frac{Du}{dx} = \frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} = 0,$$

$$\frac{Du}{dy} = \frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy} = 0,$$

we shall obtain, by the simple repetition of the preceding reasonings,

$$\frac{D^2u}{dx^2} = \frac{d^2u}{dx^2} + 2\frac{d^2u}{dxdz} \cdot \frac{dz}{dx} + \frac{d^2u}{dz^2} \cdot \frac{dz^2}{dx^2} + \frac{du}{dz} \cdot \frac{d^2z}{dx^2} = 0,$$

$$\frac{D^2u}{dy^2} = \frac{d^2u}{dy^2} + 2\frac{d^2u}{dydz} \cdot \frac{dz}{dy} + \frac{d^2u}{dz^2} \cdot \frac{dz^2}{dy^2} + \frac{du}{dz} \cdot \frac{d^2z}{dy^2} = 0,$$

$$\frac{D^2u}{dxdy} = \frac{d^2u}{dxdy} + \frac{d^2u}{dydz} \cdot \frac{dz}{dx} + \frac{d^2u}{dxdz}\frac{dz}{dy} + \frac{d^2u}{dz^2}\frac{dz}{dx}\frac{dz}{dy} + \frac{du}{dz}\frac{d^2z}{dxdy} = 0.$$

From these five equations we can determine

$$\frac{dz}{dx}, \ \frac{dz}{dy}, \ \frac{d^2z}{dx^2}, \ \frac{d^2z}{dxdy}, \ \frac{d^2z}{dy^2},$$

the partial differential coefficients of the implicit function z, in terms of the partial differential coefficients of u, and therefore in terms of the variables x, y, z.

Change of Variables.

53. Let it be proposed to change the variables of an equation

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\right) = 0 \ldots\ldots\ldots (1)$$

from x and y into two variables s and t, t being, in the transformed equation, and x in the proposed equation, the independent variable. We suppose s and t to be connected with x and y, which by virtue of the equation (1) are functional of each other, by two equations

$$\left.\begin{aligned} \phi(x, y, s, t) = 0 \\ \psi(x, y, s, t) = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (2).$$

Differentiating these equations successively n times each, supposing $\frac{d^ny}{dx^n}$ to be the differential coefficient of highest order in (1), and regarding x, y, s, as implicit functions of t, we shall get $2n$ equations which we will denote by

$$\begin{array}{ll} \phi' = 0, & \psi' = 0, \\ \phi'' = 0, & \psi'' = 0, \\ \phi''' = 0, & \psi''' = 0, \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \phi^n = 0, & \psi^n = 0. \end{array}$$

From these equations, in conjunction with the equations (2), we may obtain expressions for the $2n + 2$ quantities

$$x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \dots \frac{d^nx}{dt^n}, \qquad y, \frac{d'y}{dt}, \frac{d'^2y}{dt^2}, \frac{d'^3y}{dt^3}, \dots \frac{d'^ny}{dt^n},$$

in terms of
$$s, t, \frac{ds}{dt}, \frac{d^2s}{dt^2}, \frac{d^3s}{dt^3}, \dots \frac{d^ns}{dt^n}.$$

But, by Art. (47), we are enabled to obtain expressions for

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots \frac{d^ny}{dx^n},$$

in terms of

$$\frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \dots \frac{d^nx}{dt^n}, \qquad \frac{d'y}{dt}, \frac{d'^2y}{dt^2}, \frac{d'^3y}{dt^3}, \dots \frac{d'^ny}{dt^n}.$$

Hence we are able to obtain expressions for y and its n differential coefficients with regard to x, in terms of s, t, and the n differential coefficients of s with regard to t. The equation (1) may therefore be transformed by substitution into an equivalent one

$$F\left(s, t, \frac{ds}{dt}, \frac{d^2s}{dt^2}, \frac{d^3s}{dt^3}, \dots \frac{d^ns}{dt^n}\right) = 0.$$

Transformation of one system of independent Variables into another.

54. Let z be a function of two independent variables x and y. We propose to express the partial differential coef-

ficients of z, taken with regard to x and y, in terms of those of another function r, taken with regard to two other independent variables θ and ϕ. The six variables x, y, z, r, θ, ϕ, are supposed to be connected together by three equations

$$\left.\begin{aligned} F(x, y, z, \theta, \phi, r) = 0 \\ F_1(x, y, z, \theta, \phi, r) = 0 \\ F_2(x, y, z, \theta, \phi, r) = 0 \end{aligned}\right\} \dots\dots\dots\dots (1),$$

any four of the six variables, since z is by the supposition some function of x and y, being thus functions of the two remaining ones, which will be entirely arbitrary.

It is evident that r may be regarded either as a function of x and y alone, or as a function of θ and ϕ alone, θ and ϕ being in the latter case regarded each of them as a function of x and y alone. Hence we see that, first considering y and next x as constant,

$$\left.\begin{aligned} \frac{dr}{dx} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dx} + \frac{dr}{d\phi}\cdot\frac{d\phi}{dx} \\ \frac{dr}{dy} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dy} + \frac{dr}{d\phi}\cdot\frac{d\phi}{dy} \end{aligned}\right\} \dots\dots\dots\dots (2),$$

$\frac{dr}{dx}, \frac{dr}{dy}; \frac{d\theta}{dx}, \frac{d\theta}{dy}$; and $\frac{d\phi}{dx}, \frac{d\phi}{dy}$; being the partial differential coefficients of r, θ, and ϕ, with regard to x and y, when r, θ, and ϕ, are expressed entirely in terms of x and y; $\frac{dr}{d\theta}, \frac{dr}{d\phi}$, being the partial differential coefficients of r, with regard to θ and ϕ, when r is expressed entirely in terms of θ and ϕ.

Again, from the equations (1), we have, y being considered constant,

$$\left.\begin{aligned} \frac{DF}{dx} = \frac{dF}{d\theta}\,\frac{d\theta}{dx} + \frac{dF}{d\phi}\,\frac{d\phi}{dx} + \frac{dF}{dr}\,\frac{dr}{dx} + \frac{dF}{dx} + \frac{dF}{dz}\,\frac{dz}{dx} = 0 \\ \frac{DF_1}{dx} = \frac{dF_1}{d\theta}\,\frac{d\theta}{dx} + \frac{dF_1}{d\phi}\,\frac{d\phi}{dx} + \frac{dF_1}{dr}\,\frac{dr}{dx} + \frac{dF_1}{dx} + \frac{dF_1}{dz}\,\frac{dz}{dx} = 0 \\ \frac{DF_2}{dx} = \frac{dF_2}{d\theta}\,\frac{d\theta}{dx} + \frac{dF_2}{d\phi}\,\frac{d\phi}{dx} + \frac{dF_2}{dr}\,\frac{dr}{dx} + \frac{dF_2}{dx} + \frac{dF_2}{dz}\,\frac{dz}{dx} = 0 \end{aligned}\right\} \dots (3).$$

Considering x constant, we have from (1),

$$\left.\begin{aligned}
\frac{DF}{dy} &= \frac{dF}{d\theta}\frac{d\theta}{dy} + \frac{dF}{d\phi}\frac{d\phi}{dy} + \frac{dF}{dr}\frac{dr}{dy} + \frac{dF}{dy} + \frac{dF}{dz}\frac{dz}{dy} = 0 \\
\frac{DF_1}{dy} &= \frac{dF_1}{d\theta}\frac{d\theta}{dy} + \frac{dF_1}{d\phi}\frac{d\phi}{dy} + \frac{dF_1}{dr}\frac{dr}{dy} + \frac{dF_1}{dy} + \frac{dF_1}{dz}\frac{dz}{dy} = 0 \\
\frac{DF_2}{dy} &= \frac{dF_2}{d\theta}\frac{d\theta}{dy} + \frac{dF_2}{d\phi}\frac{d\phi}{dy} + \frac{dF_2}{dr}\frac{dr}{dy} + \frac{dF_2}{dy} + \frac{dF_2}{dz}\frac{dz}{dy} = 0
\end{aligned}\right\} \ldots (4).$$

From the former of the equations (2) combined with (3), we may determine

$$\frac{d\theta}{dx},\ \frac{d\phi}{dx},\ \frac{dr}{dx},\ \frac{dz}{dx},$$

in terms of

$$\frac{dr}{d\theta},\ \frac{dr}{d\phi}:$$

and, from the latter of equations (2) combined with (4), we may find

$$\frac{d\theta}{dy},\ \frac{d\phi}{dy},\ \frac{dr}{dy},\ \frac{dz}{dy},$$

also in terms of $\frac{dr}{d\theta}$, $\frac{dr}{d\phi}$.

The above conclusions enable us to transform an equation

$$f\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right) = 0,$$

into an equivalent one

$$f_1\left(\theta, \phi, r, \frac{dr}{d\theta}, \frac{dr}{d\phi}\right) = 0.$$

Proceeding to the second order of partial differentiation we shall have, from (2), expressions for

$$\frac{d^2r}{dx^2},\ \frac{d^2r}{dx\,dy},\ \frac{d^2r}{dy^2},$$

in terms of

$$\begin{aligned}
&\frac{dr}{d\theta},\ \frac{dr}{d\phi},\ \frac{d^2r}{d\theta^2},\ \frac{d^2r}{d\theta\,d\phi},\ \frac{d^2r}{d\phi^2}, \\
&\frac{d\theta}{dx},\ \frac{d\theta}{dy},\ \frac{d^2\theta}{dx^2},\ \frac{d^2\theta}{dx\,dy},\ \frac{d^2\theta}{dy^2}, \\
&\frac{d\phi}{dx},\ \frac{d\phi}{dy},\ \frac{d^2\phi}{dx^2},\ \frac{d^2\phi}{dx\,dy},\ \frac{d^2\phi}{dy^2},
\end{aligned}$$

From (3) and (4) we shall have nine equations,

$$\left.\begin{array}{lll}\dfrac{D^2F}{dx^2}=0, & \dfrac{D^2F_1}{dx^2}=0, & \dfrac{D^2F_2}{dx^2}=0,\\ \dfrac{D^2F}{dx\,dy}=0, & \dfrac{D^2F_1}{dx\,dy}=0, & \dfrac{D^2F_2}{dx\,dy}=0,\\ \dfrac{D^2F}{dy^2}=0, & \dfrac{D^2F_1}{dy^2}=0, & \dfrac{D^2F_2}{dy^2}=0,\end{array}\right\}\ldots\ldots (5),$$

involving

$$\frac{dz}{dx},\ \frac{dz}{dy},\ \frac{d^2z}{dx^2},\ \frac{d^2z}{dx\,dy},\ \frac{d^2z}{dy^2},$$
$$\frac{dr}{dx},\ \frac{dr}{dy},\ \frac{d^2r}{dx^2},\ \frac{d^2r}{dx\,dy},\ \frac{d^2r}{dy^2},$$
$$\frac{d\theta}{dx},\ \frac{d\theta}{dy},\ \frac{d^2\theta}{dx^2},\ \frac{d^2\theta}{dx\,dy},\ \frac{d^2\theta}{dy^2},$$
$$\frac{d\phi}{dx},\ \frac{d\phi}{dy},\ \frac{d^2\phi}{dx^2},\ \frac{d^2\phi}{dx\,dy},\ \frac{d^2\phi}{dy^2}.$$

The three equations for $\frac{d^2r}{dx^2}$, $\frac{d^2r}{dx\,dy}$, $\frac{d^2r}{dy^2}$, obtained from (2), together with the equations (5), twelve equations in all, will enable us to express the quantities

$$\frac{d^2z}{dx^2},\ \frac{d^2z}{dx\,dy},\ \frac{d^2z}{dy^2},$$
$$\frac{d^2r}{dx^2},\ \frac{d^2r}{dx\,dy},\ \frac{d^2r}{dy^2},$$
$$\frac{d^2\theta}{dx^2},\ \frac{d^2\theta}{dx\,dy},\ \frac{d^2\theta}{dy^2},$$
$$\frac{d^2\phi}{dx^2},\ \frac{d^2\phi}{dx\,dy},\ \frac{d^2\phi}{dy^2},$$

in terms of

$$\frac{dr}{d\theta},\ \frac{dr}{d\phi},\ \frac{d^2r}{d\theta^2},\ \frac{d^2r}{d\theta\,d\phi},\ \frac{d^2r}{d\phi^2},$$

and

$$\frac{dz}{dx},\ \frac{dz}{dy},\ \frac{dr}{dx},\ \frac{dr}{dy},\ \frac{d\theta}{dx},\ \frac{d\theta}{dy},\ \frac{d\phi}{dx},\ \frac{d\phi}{dy},$$

the last eight of these quantities being, as we have before shewn, expressible in terms of

$$\frac{dr}{d\theta}, \ \frac{dr}{d\phi}.$$

From the above conclusions we are enabled to transform an equation

$$f\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}, \frac{d^2z}{dx^2}, \frac{d^2z}{dx\,dy}, \frac{d^2z}{dy^2}\right) = 0,$$

into an equivalent one

$$f_1\left(\theta, \phi, r, \frac{dr}{d\theta}, \frac{dr}{d\phi}, \frac{d^2r}{d\theta^2}, \frac{d^2r}{d\theta\,d\phi}, \frac{d^2r}{d\phi^2}\right) = 0.$$

We might proceed, by a continuation of the process of successive differentiation, to the transformation of partial differential equations in x, y, z, of any order whatever to equivalent ones in θ, ϕ, r. It is easily seen also that the same method of transformation may be extended to differential equations involving any number of independent variables whatever.

Ex. Transform the differential equation

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0,$$

where x and y are the independent variables, into one in which θ and ϕ shall be independent variables, having given that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

We have, considering y constant,

$$\frac{dz}{dx} = \frac{dz}{d\theta}\frac{d\theta}{dx} + \frac{dz}{dr}\frac{dr}{dx}.$$

But, since $\qquad x^2 + y^2 = r^2, \quad \text{and} \quad \frac{y}{x} = \tan\theta,$

there is $\qquad \frac{dr}{dx} = \frac{x}{r} = \cos\theta,$

and $\qquad \frac{d\theta}{dx}\sec^2\theta = -\frac{y}{x^2}, \quad \frac{d\theta}{dx} = -\frac{y}{r^2}:$

hence
$$\frac{dz}{dx} = -\frac{y}{r^2} \cdot \frac{dz}{d\theta} + \cos\theta \cdot \frac{dz}{dr},$$

$$\frac{d^2z}{dx^2} = -\frac{y}{r^2}\left(\frac{d^2z}{d\theta^2}\frac{d\theta}{dx} + \frac{d^2z}{dr\,d\theta}\frac{dr}{dx}\right) + \frac{2y}{r^3}\frac{dr}{dx}\frac{dz}{d\theta}$$

$$- \sin\theta\frac{d\theta}{dx}\frac{dz}{dr} + \cos\theta\left(\frac{d^2z}{d\theta\,dr}\frac{d\theta}{dx} + \frac{d^2z}{dr^2}\frac{dr}{dx}\right)$$

$$= -\frac{y}{r^2}\left(-\frac{d^2z}{d\theta^2}\frac{y}{r^2} + \cos\theta\frac{d^2z}{dr\,d\theta}\right) + \frac{2y\cos\theta}{r^3}\frac{dz}{d\theta}$$

$$+ \frac{y\sin\theta}{r^2}\frac{dz}{dr} + \cos\theta\left(-\frac{y}{r^2}\frac{d^2z}{d\theta\,dr} + \cos\theta\frac{d^2z}{dr^2}\right)$$

$$= \frac{\sin^2\theta}{r^2}\frac{d^2z}{d\theta^2} - \frac{2\sin\theta\cos\theta}{r}\frac{d^2z}{dr\,d\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{dz}{d\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{dz}{dr} + \cos^2\theta\frac{d^2z}{dr^2}.$$

Putting $\frac{1}{2}\pi - \theta$ for θ, the expression for $\frac{d^2z}{dx^2}$ will be changed into that for $\frac{d^2z}{dy^2}$: hence

$$\frac{d^2z}{dy^2} = \frac{\cos^2\theta}{r^2}\frac{d^2z}{d\theta^2} + \frac{2\sin\theta\cos\theta}{r}\frac{d^2z}{dr\,d\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{dz}{d\theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{dz}{dr} + \sin^2\theta\frac{d^2z}{dr^2},$$

and therefore

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = \frac{1}{r^2}\frac{d^2z}{d\theta^2} + \frac{1}{r}\frac{dz}{dr} + \frac{d^2z}{dr^2} = 0.$$

CHAPTER IV.

ELIMINATION OF CONSTANTS AND FUNCTIONS.

Elimination of Constants.

55. Let the equation $u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

involve n constants together with two variables x and y If we differentiate this equation successively n times, we shall obtain n differential equations

$$Du = 0,\quad D^2u = 0,\quad D^3u = 0, \ldots . D^nu = 0,$$

involving the n constants, the variables x, y, and the $2n$ differentials

$$dx,\ d^2x,\ d^3x, \ldots\ldots d^nx,$$
$$dy,\ d^2y,\ d^3y, \ldots\ldots d^ny.$$

If we suppose dx to be constant, then d^2x, $d^3x \ldots . d^nx$, will disappear from the equations. We shall thus have $n + 1$ equations involving n constants; and therefore, eliminating the constants, we shall arrive at a differential equation of the n^{th} order in regard to the differentials of x and y, or, if x be the independent variable, of the n^{th} order in regard to the differentials of y.

Ex. Let $(x - a)^2 + (y - b)^2 = c^2$,

a, b, c, being constants: then

$$(x - a)\ dx + (y - b)\ dy = 0,$$
$$(x - a)\ d^2x + (y - b)\ d^2y + dx^2 + dy^2 = 0,$$
$$(x - a)\ d^3x + (y - b)\ d^3y + 3dx\, d^2x + 3dy\, d^2y = 0.$$

From the first two of these differential equations there is

$$x - a = \frac{dy\ (dx^2 + dy^2)}{dx\, d^2y - dy\, d^2x},\quad y - b = \frac{dx\ (dy^2 + dx^2)}{dy\, d^2x - dx\, d^2y},$$

and therefore, from the third,

$$(dx^2 + dy^2)(d^3x\,dy - dx\,d^3y) + 3\,(dx\,d^2x + dy\,d^2y)\,(dx\,d^2y - d^2x\,dy) = 0.$$

If x be taken as the independent variable,

$$d^2x = 0,\ d^3x = 0,$$

and the result is reduced to

$$-\,(dx^2 + dy^2)\,dx\,d^3y + 3\,dx\,dy\,(d^2y)^2 = 0,$$

or

$$\left(1 + \frac{dy^2}{dx^2}\right)\frac{d^3y}{dx^3} - 3\,\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0.$$

We may also express the constants a, b, c, in terms of the second differentials of the variables. In fact

$$a = x - \frac{dy\,(dx^2 + dy^2)}{dx\,d^2y - dy\,d^2x},\quad b = y - \frac{dx\,(dy^2 + dx^2)}{dy\,d^2x - dx\,d^2y},$$

and

$$c^2 = \frac{(dx^2 + dy^2)^3}{(dx\,d^2y - dy\,d^2x)^2}.$$

Partial elimination of the Constants.

56. Instead of differentiating the equation

$$u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

n times, suppose that we differentiate it successively only m times, m being some number less than n. Then we shall have $m + 1$ equations involving n constants: we may between these equations eliminate m constants, and shall thus obtain an equation of the m^{th} order of differentials containing $n - m$ arbitrary constants. Since the m constants which we eliminate may be chosen arbitrary, it is evident that we may form as many equations of the order m, containing $n - m$ constants, as there are combinations of n things taken m at a time: we may therefore obtain of such equations a number

$$\frac{n\,(n-1)\,(n-2)\ldots.(n-m+1)}{1.2.3.\ldots m}.$$

If we differentiate any one of these differential equations $n - m$ times in succession, we shall have altogether $n - m + 1$ differential equations involving $n - m$ constants. By the elimination of these $n - m$ constants we shall arrive at a differential equation of the n^{th} order. It is important to remark that this final differential equation will always coincide identically with that obtained by the process of Art. (55). That such must be the case will be evident when it is considered that neither method of elimination involves any limitation of the generality of the variables and their differentials, and that accordingly the results must in both cases be perfectly general.

Ex. Let $$(x - a)^2 + (y - b)^2 = c^2:$$

then $$(x - a)\,dx + (y - b)\,dy = 0 \,\ldots\ldots\ldots\ldots (1),$$

$$(x - a)\,d^2x + (y - b)\,d^2y + dx^2 + dy^2 = 0 \,\ldots\ldots (2).$$

Eliminating a between (1) and (2), we have

$$(y - b)(dx\,d^2y - d^2x\,dy) + dx\,(dx^2 + dy^2) = 0 \ldots\ldots (3).$$

Differentiating (3), we have

$$(y - b)(dx\,d^3y - d^3x\,dy) + dy\,(dx\,d^2y - d^2x\,dy) + d^2x\,(dx^2 + dy^2)$$
$$+ 2dx\,(dx\,d^2x + dy\,d^2y) = 0,$$

or $$(y - b)\,(dx\,d^3y - d^3x\,dy) + 3dx\,(dx\,d^2x + dy\,d^2y) = 0 \ldots (4).$$

Eliminating b between (3) and (4), we get

$$(dx^2 + dy^2)\,(d^3x\,dy - dx\,d^3y) + 3\,(dx\,d^2x + dy\,d^2y)\,(dx\,d^2y - d^2x\,dy) = 0,$$

a result coinciding with that obtained by the method of Art. (55).

Elimination of irrational, logarithmic, exponential, and circular Functions of known Functions.

57. Let $$u = f(x, y, c_1, c_2, c_3, \ldots c_n) = 0$$

be an equation between two variables x and y; where $c_1, c_2, c_3, \ldots c_n$, are n irrational, logarithmic, exponential, or circular functions of $s_1, s_2, s_3, \ldots s_n$, respectively, $s_1, s_2, s_3, \ldots s_n$, being known functions of x and y. If we differentiate this equation successively n times, we shall obtain n differential equations

$$Du = 0, \quad D^2u = 0, \quad D^3u = 0, \ldots\ldots D^nu = 0,$$

involving the differentials of x and y up to the n^{th} order, together with the n^n expressions

$$\begin{array}{l}
\dfrac{dc_1}{ds_1}, \; \dfrac{d^2c_1}{ds_1^{\,2}}, \; \dfrac{d^3c_1}{ds_1^{\,3}}, \ldots \; \dfrac{d^nc_1}{ds_1^{\,n}}, \\[2ex]
\dfrac{dc_2}{ds_2}, \; \dfrac{d^2c_2}{ds_2^{\,2}}, \; \dfrac{d^3c_2}{ds_2^{\,3}}, \ldots \; \dfrac{d^nc_2}{ds_2^{\,n}}, \\[2ex]
\dfrac{dc_3}{ds_3}, \; \dfrac{d^2c_3}{ds_3^{\,2}}, \; \dfrac{d^3c_3}{ds_3^{\,3}}, \ldots \; \dfrac{d^nc_3}{ds_3^{\,n}}, \\
\cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\
\cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\
\dfrac{dc_n}{ds_n}, \; \dfrac{d^2c_n}{ds^2_n}, \; \dfrac{d^3c_n}{ds_n^{\,3}}, \ldots \; \dfrac{d^nc_n}{ds_n^{\,n}}.
\end{array}$$

Now the first rank of these expressions can be expressed in terms of c_1, the second in terms of c_2, the third in terms of c_3, and so on: hence we shall have, in all, $n+1$ equations containing the n functions $c_1, c_2, c_3, \ldots c_n$. We may therefore eliminate these functions, and shall thus arrive at a differential equation of the n^{th} order in x and y.

Ex. 1. To eliminate the functions

$$\left(\frac{x}{a}\right)^m, \; \left(\frac{x}{a}\right)^n,$$

from the equation

$$y = \left(\frac{x}{a}\right)^m + \left(\frac{x}{a}\right)^n \ldots\ldots\ldots\ldots\ldots (1),$$

where m and n may be supposed to denote any numerical fractions.

Differentiating (1) we obtain, dx being considered constant,

$$\frac{dy}{dx} = \frac{m}{a^m}\, x^{m-1} + \frac{n}{a^m}\, x^{n-1},$$

$$\frac{d^2y}{dx^2} = \frac{m(m-1)}{a^m}\, x^{m-2} + \frac{n(n-1)}{a^n}\, x^{n-2}:$$

whence

$$x\,\frac{dy}{dx} = m\left(\frac{x}{a}\right)^m + n\left(\frac{x}{a}\right)^n \ldots\ldots\ldots\ldots\ldots (2),$$

$$x^2\,\frac{d^2y}{dx^2} = m(m-1)\left(\frac{x}{a}\right)^m + n(n-1)\left(\frac{x}{a}\right)^n \ldots (3).$$

From (1) and (2) we have

$$ny - x\frac{dy}{dx} = (n - m)\left(\frac{x}{a}\right)^m;$$

also, from (1) and (3),

$$n(n-1)y - x^2\frac{d^2y}{dx^2} = \{n(n-1) - m(m-1)\}\left(\frac{x}{a}\right)^m$$

$$= (n-m)(m+n-1)\left(\frac{x}{a}\right)^m:$$

from these last two equations we get

$$n(n-1)y - x^2\frac{d^2y}{dx^2} = (m+n-1)\left(ny - x\frac{dy}{dx}\right),$$

or $$mny - (m+n-1)x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0,$$

a differential equation of the second order.

Ex. 2. To eliminate log x from the equation

$$y = x \log x.$$

Differentiating, we have

$$\frac{dy}{dx} = \log x + 1,$$

whence $$x\frac{dy}{dx} = x\log x + x = y + x.$$

Ex. 3. To eliminate sin $(x + y)$ from the equation

$$y = \sin(x + y).$$

Putting $\frac{dy}{dx} = y'$, for the sake of brevity, we get, by differentiation, $$y' = \cos(x+y) \,.\, (1 + y'),$$

or $$\frac{y'^2}{(1+y')^2} = \cos^2(x+y).$$

But, from the proposed equation,

$$y^2 = \sin^2(x+y):$$

hence $$y^2 + \frac{y'^2}{(1+y')^2} = 1,$$

$$y^2(1+y')^2 + y'^2 = (1+y')^2.$$

Ex. 4. To eliminate the exponential functions from the equation
$$ae^{y} + be^{-y} = fe^{x} + ge^{-x}.$$

Differentiating we get
$$(ae^{y} - be^{-y})\, y' = fe^{x} - ge^{-x},$$
$$(ae^{y} - be^{-y})\, y'' + (ae^{y} + be^{-y})\, y'^{2} = fe^{x} + ge^{-x};$$
whence $(fe^{x} - ge^{-x})\, y'' + y'^{3}\, (fe^{x} + ge^{-x}) = y'\, (fe^{x} + ge^{-x})\ \ldots\ (1).$

Differentiating (1),
$$(fe^{x} + ge^{-x})\, y'' + (fe^{x} - ge^{-x})\, y''' + 3y'^{2}y''\, (fe^{x} + ge^{-x}) + y'^{3}\, (fe^{x} - ge^{-x})$$
$$= (fe^{x} - ge^{-x})\, y' + (fe^{x} + ge^{-x})\, y'',$$
or
$$3y'^{2}y''\, (fe^{x} + ge^{-x}) = (y' - y'^{3} - y''')\, (fe^{x} - ge^{-x}):$$
but, from (1),
$$y''\, (fe^{x} - ge^{-x}) = (y' - y'^{3})\, (fe^{x} + ge^{-x}):$$
multiplying together these last two equations, we get
$$3y'y''^{2} = (1 - y'^{2})\, (y' - y'^{3} - y''').$$

Elimination of an arbitrary Function of a known Function.

58. Suppose that
$$u = \phi(v),$$
where u, v, are known functions of three variables x, y, z, and $\phi(v)$ a perfectly arbitrary function of v.

Differentiating this equation, first with regard to x and next with regard to y, we have
$$\frac{Du}{dx} = \phi'(v) \cdot \frac{Dv}{dx}, \quad \frac{Du}{dy} = \phi'(v) \cdot \frac{Dv}{dy},$$
and therefore
$$\frac{Du}{dx} \cdot \frac{Dv}{dy} = \frac{Du}{dy} \cdot \frac{Dv}{dx},$$
or
$$\left(\frac{du}{dx} + \frac{du}{dz}\,\frac{dz}{dx}\right) \cdot \left(\frac{dv}{dy} + \frac{dv}{dz}\,\frac{dz}{dy}\right)$$
$$= \left(\frac{du}{dy} + \frac{du}{dz}\,\frac{dz}{dy}\right) \cdot \left(\frac{dv}{dx} + \frac{dv}{dz}\,\frac{dz}{dx}\right),$$

or $$\left(\frac{du}{dz}\cdot\frac{dv}{dy}-\frac{du}{dy}\cdot\frac{dv}{dz}\right)\cdot\frac{dz}{dx}+\left(\frac{dv}{dz}\cdot\frac{du}{dx}-\frac{dv}{dx}\cdot\frac{du}{dz}\right)\frac{dz}{dy}$$
$$=\frac{du}{dy}\cdot\frac{dv}{dx}-\frac{du}{dx}\cdot\frac{dv}{dy}.$$

Thus we see that, although the equation $u=\phi(v)$ varies in an infinite number of ways, with the variation in the form of the function $\phi(v)$ of v, yet that all this family of equations possesses one common partial differential equation.

Ex. Eliminate the arbitrary function from the equation

$$\frac{z-c}{y-b}=\phi\left(\frac{z-c}{x-a}\right).$$

Differentiating, first with regard to x and then with regard to y, we have

$$\frac{1}{y-b}\cdot\frac{dz}{dx}=\phi'\left(\frac{z-c}{x-a}\right)\cdot\frac{D}{dx}\left(\frac{z-c}{x-a}\right)$$
$$=\phi'\left(\frac{z-c}{x-a}\right)\cdot\frac{1}{(x-a)^2}\cdot\left\{(x-a)\frac{dz}{dx}-(z-c)\right\},$$

and $$\frac{1}{(y-b)^2}\left\{(y-b)\frac{dz}{dy}-(z-c)\right\}=\phi'\left(\frac{z-c}{x-a}\right)\cdot\frac{1}{x-a}\cdot\frac{dz}{dy}.$$

Eliminating $\phi'\left(\frac{z-c}{x-a}\right)$ between these two equations, we see that

$$\frac{x-a}{y-b}\cdot\frac{dz}{dx}\cdot\frac{dz}{dy}=\frac{1}{(y-b)^2}\left\{(y-b)\frac{dz}{dy}-(z-c)\right\}\cdot\left\{(x-a)\frac{dz}{dx}-(z-c)\right\},$$

whence $$(x-a)\frac{dz}{dx}+(y-b)\frac{dz}{dy}=z-c.$$

59. If the arbitrary function $\phi(v)$ involves y only, then it is sufficient to differentiate with respect to x: in fact

$$\frac{Du}{dx}=0,\qquad\text{or}\quad\frac{du}{dx}+\frac{du}{dz}\,\frac{dz}{dx}=0,$$

the arbitrary function being thus eliminated by one differentiation.

Ex. Let $z = xy\,\phi(y)$, whence

$$\frac{z}{x} = y\,\phi(y):$$

then $$\frac{1}{x^2}\left(x\frac{dz}{dx} - z\right) = 0, \qquad \text{or } x\frac{dz}{dx} = z.$$

Elimination of any number of arbitrary Functions of known Functions.

60. Let there be an equation

$$u = f\{x, y, z, \phi_1(c_1), \phi_2(c_2), \phi_3(c_3), \ldots\ldots \phi_m(c_m)\} = 0,$$

involving m arbitrary functions

$$\phi_1(c_1), \phi_2(c_2), \phi_3(c_3), \ldots\ldots \phi_m(c_m),$$

where $c_1, c_2, c_3, \ldots\ldots c_m$, are all known functions of x, y, z.

If we differentiate the proposed equation n times we shall have, in all, the following equations,

$$u = 0,$$

$$\frac{Du}{dx} = 0, \quad \frac{Du}{dy} = 0,$$

$$\frac{D^2u}{dx^2} = 0, \quad \frac{D^2u}{dxdy} = 0, \quad \frac{D^2u}{dy^2} = 0,$$

$$\frac{D^3u}{dx^3} = 0, \quad \frac{D^3u}{dx^2dy} = 0, \quad \frac{D^3u}{dxdy^2} = 0, \quad \frac{D^3u}{dy^3} = 0.$$

. . . .

. . . .

$$\frac{D^nu}{dx^n} = 0, \frac{D^nu}{dx^{n-1}dy} = 0, \frac{D^nu}{dx^{n-2}dy^2} = 0, \ldots \frac{D^nu}{dx^2dy^{n-2}} = 0, \frac{D^nu}{dxdy^{n-1}} = 0, \frac{D^nu}{dy^n} = 0,$$

the number of which is

$$1 + 2 + 3 + \ldots\ldots + (n+1) = \tfrac{1}{2}(n+1)(n+2).$$

These equations will involve the following functions,

$$\begin{array}{lllll}
\phi_1(c_1), & \phi_2(c_2), & \phi_3(c_3), & \ldots\ldots & \phi_m(c_m), \\
\phi_1'(c_1), & \phi_2'(c_2), & \phi_3'(c_3), & \ldots\ldots & \phi_m'(c_m), \\
\phi_1''(c_1), & \phi_2''(c_2), & \phi_3''(c_3), & \ldots\ldots & \phi_m''(c_m), \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\phi_1^n(c_1), & \phi_2^n(c_2), & \phi_3^n(c_3), & \ldots\ldots & \phi_m^n(c_m),
\end{array}$$

the number of which is $m(n+1)$. We have therefore $m(n+1)$ functions and $\frac{1}{2}(n+1)(n+2)$ equations: in order to eliminate these functions it is sufficient that

$$\frac{1}{2}(n+1)(n+2) > m(n+1),$$

$$n + 2 > 2m,$$

$$n > 2m - 2,$$

or

$$n = 2m - 1.$$

The number of quantities to be eliminated will therefore be

$$m(n+1) = 2m^2;$$

and the number of equations involving them

$$\frac{1}{2}(n+1)(n+2) = \frac{1}{2} \cdot 2m(2m+1) = 2m^2 + m:$$

we shall therefore arrive at, as the result of our elimination, m partial differential equations between the variables x, y, z, of the order $2m - 1$.

If $m = 1$, then $2m - 1 = 1$; if $m = 2$, then $2m - 1 = 3$; if $m = 3$, then $2m - 1 = 5$, and so on. That is, if there be one arbitrary function in the proposed equation, there will be one final equation of the first order; if two functions, two final equations of the third order; if three functions, three final equations of the fifth order, and so on.

This is the general theory of such eliminations: it frequently happens however, for particular forms of the proposed equation, that the elimination may be effected without proceeding to so high an order of differentiation, and arriving at so many final equations, as would be implied by these general considerations. So that we must consider the general theory as defining the number of sufficient, but not in all cases the number of necessary differentiations.

Ex. 1. Eliminate the arbitrary functions from the equation

$$z = x\,\phi(z) + y\,\chi(z).$$

Putting $$\frac{\phi(z)}{z} = \phi_1(z), \quad \frac{\chi(z)}{z} = \chi_1(z),$$

we have $$1 = x\phi_1(z) + y\chi_1(z):$$

hence $$0 = \{x\phi_1'(z) + y\chi_1'(z)\}\frac{dz}{dx} + \phi_1(z),$$

$$0 = \{x\phi_1'(z) + y\chi_1'(z)\}\frac{dz}{dy} + \chi_1(z):$$

and therefore, putting $\phi_1(z) = \chi_1(z)\,.\,f(z)$,

$$\frac{\dfrac{dz}{dx}}{\dfrac{dz}{dy}} = \frac{\phi_1(z)}{\chi_1(z)} = f(z):$$

differentiating this equation, first with regard to x and next with regard to y, we get

$$\frac{d^2z}{dx^2}\cdot\frac{dz}{dy} - \frac{dz}{dx}\cdot\frac{d^2z}{dxdy} = f'(z)\cdot\frac{dz}{dx}\cdot\left(\frac{dz}{dy}\right)^2,$$

$$\frac{d^2z}{dxdy}\cdot\frac{dz}{dy} - \frac{dz}{dx}\cdot\frac{d^2z}{dy^2} = f'(z)\cdot\left(\frac{dz}{dy}\right)^3:$$

eliminating $f'(z)$, we have

$$\left(\frac{dz}{dy}\right)^2\cdot\frac{d^2z}{dx^2} - 2\,\frac{dz}{dx}\,\frac{dz}{dy}\,\frac{d^2z}{dxdy} + \left(\frac{dz}{dx}\right)^2\cdot\frac{d^2z}{dy^2} = 0.$$

Thus we see that, instead of two final equations of the third order, we have a single equation of the second order.

Ex. 2. Suppose that

$$z = \phi\,(x + y) + xy\,\psi\,(x - y).$$

The elimination of the arbitrary functions in this case can be effected only by proceeding to partial differentials of the third order, there being two final equations of this order, a result in harmony with the general theory which has been laid down. For the discussion of this example the student is referred to Lacroix, *Traité du Calcul Différentiel*, tom. I. p. 234.

Ex. 3. To eliminate the arbitrary functions from the equation

$$z = f\left(\frac{x^2+y^2}{x^2-y^2}\right) + F\left(\frac{x^2-y^2}{x^2+y^2}\right).$$

The two functions are readily reduced to one: thus

$$f\left(\frac{x^2+y^2}{x^2-y^2}\right) = f\left\{\frac{1+\dfrac{y^2}{x^2}}{1-\dfrac{y^2}{x^2}}\right\} = \text{a function of } \frac{y}{x},$$

and $$F\left(\frac{x^2-y^2}{x^2+y^2}\right) = F\left\{\frac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}\right\} = \text{a function of } \frac{y}{x}:$$

hence $$z = \phi\left(\frac{y}{x}\right),$$

ϕ being a symbol of arbitrary functionality. Differentiating, first with regard to x and then with regard to y, we have

$$\frac{dz}{dx} = -\phi'\left(\frac{y}{x}\right).\frac{y}{x^2},$$

$$\frac{dz}{dy} = \phi'\left(\frac{y}{x}\right).\frac{1}{x},$$

and therefore $$x\frac{dz}{dx} + y\frac{dz}{dy} = 0.$$

This result would have been obtained by differentiating the proposed equation, without modification: the operation would however have been more tedious.

Elimination of arbitrary Functions of unknown Functions.

61. Suppose that we have two equations

$$u = f\{x, y, z, c, \phi(c), \chi(c), \ldots\ldots\} = 0,$$
$$v = F\{x, y, z, c, \phi(c), \chi(c), \ldots\ldots\} = 0;$$

c and z being therefore implicit functions of x and y. The functions $\phi(c)$, $\chi(c)$, ... are supposed to be m arbitrary functions of c; whence it follows that c is an arbitrary function of x and y: for, supposing z to be eliminated between the two equations, we shall obtain an equation between x, y, c, of which the form is arbitrary. We propose to eliminate by differentiation the m arbitrary functions of c and the function c itself.

If we differentiate the proposed equations n times successively, we shall obtain the following equations,

$$\left.\begin{array}{ll} \dfrac{Du}{dx} = 0, & \dfrac{Du}{dy} = 0 \\ \dfrac{Dv}{dx} = 0, & \dfrac{Dv}{dy} = 0 \end{array}\right\},$$

$$\left.\begin{array}{lll} \dfrac{D^2u}{dx^2} = 0, & \dfrac{D^2u}{dxdy} = 0, & \dfrac{D^2u}{dy^2} = 0 \\ \dfrac{D^2v}{dx^2} = 0, & \dfrac{D^2v}{dxdy} = 0, & \dfrac{D^2v}{dy^2} = 0 \end{array}\right\},$$

$$\left.\begin{array}{llll} \dfrac{D^3u}{dx^3} = 0, & \dfrac{D^3u}{dx^2dy} = 0, & \dfrac{D^3u}{dxdy^2} = 0, & \dfrac{D^3u}{dy^3} = 0 \\ \dfrac{D^3v}{dx^3} = 0, & \dfrac{D^3v}{dx^2dy} = 0, & \dfrac{D^3v}{dxdy^2} = 0, & \dfrac{D^3v}{dy^3} = 0 \end{array}\right\},$$

$$\begin{array}{ccccccccc} . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{array}$$

the whole number of these equations, together with the two original equations, being

$$2\left\{1 + 2 + 3 + \ldots\ldots + (n+1)\right\} = (n+1)(n+2).$$

These $(n+1)(n+2)$ equations will involve the quantities

$$c,$$

$$\frac{dc}{dx},\quad \frac{dc}{dy},$$

$$\frac{d^2c}{dx^2},\quad \frac{d^2c}{dxdy},\quad \frac{d^2c}{dy^2},$$

$$\frac{d^3c}{dx^3},\quad \frac{d^3c}{dx^2dy},\quad \frac{d^3c}{dxdy^2},\quad \frac{d^3c}{dy^3},$$

$$\begin{array}{cccccccc} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{array}$$

$$\frac{d^nc}{dx^n},\quad \frac{d^nc}{dx^{n-1}dy},\quad \frac{d^nc}{dx^{n-2}dy^2},\ldots\ldots\frac{d^nc}{dx^2dy^{n-2}},\quad \frac{d^nc}{dxdy^{n-1}},\quad \frac{d^nc}{dy^n},$$

the number of which is

$$1 + 2 + 3 + \ldots\ldots + (n + 1) = \tfrac{1}{2}(n + 1)(n + 2),$$

and also the functions

$$\begin{array}{llll} \phi(c), & \phi'(c), & \phi''(c), \ldots\ldots & \phi^n(c), \\ \chi(c), & \chi'(c), & \chi''(c), \ldots\ldots & \chi^n(c), \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{array}$$

of which the number is $m\,(n + 1)$.

Thus we shall have $(n + 1)\,(n + 2)$ equations, between which it is proposed to eliminate quantities of which the number is equal to

$$\tfrac{1}{2}(n + 1)(n + 2) + m\,(n + 1):$$

for this purpose it is sufficient that

$$(n + 1)(n + 2) > \tfrac{1}{2}(n + 1)(n + 2) + m\,(n + 1),$$

$$\tfrac{1}{2}(n + 1)(n + 2) > m\,(n + 1),$$

$$\tfrac{1}{2}(n + 2) > m,$$

$$n > 2m - 2;$$

or that

$$n = 2m - 1.$$

We shall have then, for the number of the equations,

$$(n + 1)(n + 2) = 2m\,(2m + 1),$$

and, for the number of the quantities to be eliminated,

$$\tfrac{1}{2}(n + 1)(n + 2) + m\,(n + 1) = m\,(2m + 1) + 2m^2 = 4m^2 + m.$$

It follows that, when the elimination is completed, we shall arrive at m partial differential equations, of the $(2m - 1)^{\text{th}}$ order, which will be satisfied by all the equations which are comprehended under the general forms of the two proposed equations.

It frequently happens however that the order of the partial differential equations necessary for the elimination of the $m + 1$ quantities c, $\phi(c)$, $\chi(c)$, ... is lower than the $(2m - 1)^{\text{th}}$. Suppose for example that there are three arbitrary functions $\phi(c)$, $\chi(c)$, $\psi(c)$: in this case $m = 3$, $n = 2m - 1 = 5$: we see then that generally to effect the proposed elimination we should have to proceed as far as the partial differentials of z of the fifth order, and should arrive at three partial differential equations of this order, between x, y, z. But if we establish the relations

$$\chi(c) = \phi'(c),\ \psi(c) = \phi''(c),$$

that is, if the equations are of the form

$$f\{x, y, z, c, \phi(c), \phi'(c), \phi''(c)\} = 0,$$
$$F\{x, y, z, c, \phi(c), \phi'(c), \phi''(c)\} = 0:$$

then, if we proceed as far as differentials of the second order, we shall have, in all, twelve equations, between which we may eliminate the eleven quantities

$$c,\ \frac{dc}{dx},\ \frac{dc}{dy},\ \frac{d^2c}{dx^2},\ \frac{d^2c}{dxdy},\ \frac{d^2c}{dy^2},$$
$$\phi(c),\ \phi'(c),\ \phi''(c),\ \phi'''(c),\ \phi''''(c),$$

the result being a single equation between x, y, z, involving partial differentials of z only to the second order.

Ex. Eliminate the arbitrary functions from the equations

$$x\,\phi(c) + y\,\chi(c) + z\,\psi(c) = 1 \ldots\ldots\ldots\ldots (1),$$
$$x\,\phi'(c) + y\,\chi'(c) + z\,\psi'(c) = 0 \ldots\ldots\ldots\ldots (2).$$

Differentiating (1), with respect to x, we have

$$\{x\,\phi'(c) + y\,\chi'(c) + z\,\psi'(c)\}\,\frac{dc}{dx} + \phi(c) + \frac{dz}{dx}\,\psi(c) = 0,$$

whence, by the aid of (2),

$$\phi(c) + \frac{dz}{dx}\,\psi(c) = 0 \ldots\ldots\ldots\ldots (3).$$

Similarly, differentiating (1) with respect to y,

$$\chi(c) + \frac{dz}{dy}\,\psi(c) = 0 \ldots\ldots\ldots\ldots (4).$$

From (3) and (4), putting

$$\frac{\phi(c)}{\psi(c)} = f(c),\ \frac{\chi(c)}{\psi(c)} = F(c),$$

we have
$$f(c)+\frac{dz}{dx}=0 \;\dots\dots\dots\dots\dots (5),$$

$$F(c)+\frac{dz}{dy}=0 \;\dots\dots\dots\dots\dots (6).$$

From (5), we get
$$f'(c)\frac{dc}{dx}+\frac{d^2z}{dx^2}=0,$$

$$f'(c)\frac{dc}{dy}+\frac{d^2z}{dxdy}=0,$$

and therefore
$$\frac{dc}{dy}\cdot\frac{d^2z}{dx^2}=\frac{dc}{dx}\cdot\frac{d^2z}{dxdy} \;\dots\dots\dots\dots (7).$$

In like manner, from (6),

$$\frac{dc}{dx}\cdot\frac{d^2z}{dy^2}=\frac{dc}{dy}\cdot\frac{d^2z}{dxdy} \;\dots\dots\dots\dots (8).$$

From (7) and (8), we get, as our final result,

$$\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}=\left(\frac{d^2z}{dxdy}\right)^2.$$

Elimination of arbitrary Functions when the number of independent Variables exceeds two.

62. In the preceding considerations respecting the elimination of arbitrary functions, we have always supposed that there are only two independent variables. We will now proceed to develop the theory of elimination when the number of independent variables is any whatever. For the sake of simplicity we shall confine ourselves to the case where it is not necessary to proceed to partial differentials beyond the first order.

Let
$$f(x, y, z, \dots\dots u, c)=0,$$

$x, y, z, \dots\dots$ being $(m+1)$ independent variables, and u the dependent variable. The quantity c is supposed to be an arbitrary function of $\alpha, \beta, \gamma, \dots$ which are m known functions of $x, y, z, \dots$ and u.

Taking partial differentials with regard to $x, y, z, \ldots\ldots$ in succession, we get

$$\frac{Df}{dx} = \frac{df}{dx} + \frac{df}{du}\,\frac{du}{dx} + \frac{df}{dc}\left(\frac{dc}{d\alpha}\,\frac{d\alpha}{dx} + \frac{dc}{d\beta}\,\frac{d\beta}{dx} + \frac{dc}{d\gamma}\,\frac{d\gamma}{dx} + \ldots\right) = 0,$$

$$\frac{Df}{dy} = \frac{df}{dy} + \frac{df}{du}\,\frac{du}{dy} + \frac{df}{dc}\left(\frac{dc}{d\alpha}\,\frac{d\alpha}{dy} + \frac{dc}{d\beta}\,\frac{d\beta}{dy} + \frac{dc}{d\gamma}\,\frac{d\gamma}{dy} + \ldots\right) = 0,$$

$$\frac{Df}{dz} = \frac{df}{dz} + \frac{df}{du}\,\frac{du}{dz} + \frac{df}{dc}\left(\frac{dc}{d\alpha}\,\frac{d\alpha}{dz} + \frac{dc}{d\beta}\,\frac{d\beta}{dz} + \frac{dc}{d\gamma}\,\frac{d\gamma}{dz} + \ldots\right) = 0,$$

. . . .

. . . .

Taking these differential equations conjointly with the proposed equation, we shall have altogether $m+2$ equations involving the $m+1$ arbitrary functions

$$c,\ \frac{dc}{d\alpha},\ \frac{dc}{d\beta},\ \frac{dc}{d\gamma},\ldots\ldots$$

These arbitrary functions may therefore be eliminated, when we shall arrive at a partial differential equation of the first order of partial differentials.

Ex. To eliminate the arbitrary function from the equation

$$(m-1)\,u + \frac{xy}{z} = x^m\phi\left(\frac{y}{x},\ \frac{z}{x}\right).$$

Putting $$\frac{y}{x} = \alpha,\quad \frac{z}{x} = \beta,$$

we have $$(m-1)\,u + \frac{xy}{z} = x^m\,\phi(\alpha,\beta),$$

whence

$$(m-1)\frac{du}{dx} + \frac{y}{z} = mx^{m-1}\,\phi\,(\alpha,\beta)$$
$$+\, x^m\left\{\frac{d}{d\alpha}\,\phi\,(\alpha,\beta)\,.\,\frac{-y}{x^2} + \frac{d}{d\beta}\,\phi(\alpha,\beta)\,.\,\frac{-z}{x^2}\right\},$$

$$(m-1)\,\frac{du}{dy} + \frac{x}{z} = x^m\,\frac{d}{d\alpha}\,\phi\,(\alpha,\beta)\,.\,\frac{1}{x},$$

$$(m-1)\,\frac{du}{dz} - \frac{xy}{z^2} = x^m\,\frac{d}{d\beta}\,\phi\,(\alpha,\beta)\,.\,\frac{1}{x}.$$

Multiplying these differential equations in order by x, y, z, and adding, we get, attending to the proposed equation,

$$(m-1)\left(x\frac{du}{dx}+y\frac{du}{dy}+z\frac{du}{dz}\right)+\frac{xy}{z}=mx^m\phi(a,\beta)$$

$$=m(m-1)u+\frac{mxy}{z},$$

whence $$x\frac{du}{dx}+y\frac{du}{dy}+z\frac{du}{dz}=mu+\frac{xy}{z}.$$

CHAPTER V.

EVALUATION OF INDETERMINATE FUNCTIONS.

Indeterminateness of explicit Functions of a single Variable.

63. Suppose that $$\phi(x) = \frac{f(x)}{F(x)},$$

and that, when a particular value x_0 is assigned to x, $f(x)$ and $F(x)$ both become zero: the value of $\phi(x)$ will, for such a value of x, present itself under the indeterminate form $\frac{0}{0}$. We proceed to investigate a rule which is often useful for the determination of the true value of $\phi(x_0)$.

We have generally

$$\phi(x + \delta x) = \frac{f(x) + \delta f(x)}{F(x) + \delta F(x)},$$

and, for the particular value of x_0 of x,

$$\phi(x_0 + \delta x_0) = \frac{\delta f(x)}{\delta F(x)} = \frac{\dfrac{\delta f(x)}{\delta x}}{\dfrac{\delta F(x)}{\delta x}},$$

x and δx being supposed to be replaced by x_0 and δx_0 in the expressions for

$$\frac{\delta f(x)}{\delta x}, \quad \frac{\delta F(x)}{\delta x},$$

which are generally functions of x and δx. Passing to the limit, when δx_0 becomes less than any assignable quantity, we have

$$\phi(x_0) = \frac{f'(x)}{F'(x)},$$

it being supposed that, in the expressions for the functions $f'(x)$

and $F'(x)$, x_0 is substituted for x. In other words, for any value of x which makes $f(x) = 0$ and $F(x) = 0$, the value of

$$\frac{f(x)}{F(x)},$$

is the same as that of $$\frac{f'(x)}{F'(x)}.$$

If, for this same value of x, $f'(x) = 0$ and $F'(x) = 0$; then, by the application of the same principle, we see that, for this value of x,

$$\frac{f'(x)}{F'(x)} = \frac{f''(x)}{F''(x)},$$

or $$\frac{f(x)}{F(x)} = \frac{f''(x)}{F''(x)},$$

and so on. If $f^n(x)$ and $F^n(x)$ are the lowest differential coefficients of $f(x)$ and $F(x)$, of which both do not vanish for the particular value x_0 of x, then the true value of $\phi(x_0)$ will be

$$\frac{f^n(x_0)}{F^n(x_0)},$$

an expression representing the value of

$$\frac{f^n(x)}{F^n(x)}, \text{ when } x = x_0.$$

Ex. 1. To find the value of

$$\phi(x) = \frac{a^x - b^x}{x}, \text{ when } x = 0.$$

Here $$f(x) = a^x - b^x,$$
$$\phi(x) = x:$$

hence

$f'(x) = \log a \,.\, a^x - \log b \,.\, b^x = \log a - \log b$, when $x = 0$:

$\phi'(x) = 1$:

thus $$\frac{f'(0)}{F'(0)} = \log \frac{a}{b},$$

or $$\phi(0) = \log \frac{a}{b}.$$

Ex. 2. To find the value of

$$\phi(x) = \frac{x - \sin x}{x^3}, \text{ when } x = 0.$$

Here
$$\begin{aligned} f(x) &= x - \sin x, \\ F(x) &= x^3: \end{aligned}$$

hence
$$\begin{aligned} f'(x) &= 1 - \cos x = 0, \\ F'(x) &= 3x^2 = 0: \\ f''(x) &= \sin x = 0, \\ F''(x) &= 6x = 0: \\ f'''(x) &= \cos x = 1, \\ F'''(x) &= 6. \end{aligned}$$

Hence
$$\phi(0) = \frac{f'''(0)}{F'''(0)} = \frac{1}{6}.$$

Ex. 3. To find the value of

$$\phi(x) = (1 - x) \tan \frac{\pi x}{2}, \text{ when } x = 1.$$

Putting the equation under the form

$$\phi(x) = \frac{(1 - x) \sin \frac{\pi x}{2}}{\cos \frac{\pi x}{2}},$$

we see that
$$\begin{aligned} f(x) &= (1 - x) \sin \frac{\pi x}{2}, \\ F(x) &= \cos \frac{\pi x}{2}: \end{aligned}$$

hence
$$\begin{aligned} f'(x) &= - \sin \frac{\pi x}{2} + \frac{\pi}{2} (1 - x) \cos \frac{\pi x}{2} = - 1, \\ F'(x) &= - \frac{\pi}{2} \sin \frac{\pi x}{2} = - \frac{\pi}{2}; \end{aligned}$$

and therefore
$$\phi(1) = \frac{f'(1)}{F'(1)} = \frac{2}{\pi}.$$

It would have been however more simple, observing that $\sin\frac{\pi x}{2} = 1$, when $x = 1$, to have sought the value of

$$\phi(x) = \frac{1 - x}{\cos\frac{\pi x}{2}},$$

instead of $$\phi(x) = \frac{(1 - x)\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}}.$$

We should then have

$$f(x) = 1 - x,$$

$$F(x) = \cos\frac{\pi x}{2};$$

$$f'(x) = -1,$$

$$F'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2} = -\frac{\pi}{2},$$

and, as before, $$\phi(1) = \frac{2}{\pi}.$$

Ex. 4. To evaluate

$$\phi(x) = (a^2 - x^2)^{\frac{1}{2}}.\cot\left\{\frac{\pi}{2}\left(\frac{a - x}{a + x}\right)^{\frac{1}{2}}\right\}, \text{ when } x = a.$$

By an obvious transformation we have

$$\phi(x) = (a + x)^{\frac{1}{2}}.\cos\left\{\frac{\pi}{2}\left(\frac{a - x}{a + x}\right)^{\frac{1}{2}}\right\}.\frac{(a - x)^{\frac{1}{2}}}{\sin\left\{\frac{\pi}{2}\left(\frac{a - x}{a + x}\right)^{\frac{1}{2}}\right\}}$$

$$= (2a)^{\frac{1}{2}}.\frac{(a - x)^{\frac{1}{2}}}{\sin\left\{\frac{\pi}{2}\left(\frac{a - x}{a + x}\right)^{\frac{1}{2}}\right\}}, \text{ when } x = a.$$

Put $$f(x) = (a - x)^{\frac{1}{2}},$$

$$F(x) = \sin\left\{\frac{\pi}{2}\left(\frac{a - x}{a + x}\right)^{\frac{1}{2}}\right\};$$

$f(x)$ and $F(x)$ being both zero when $x = a$.

Then, when $x = a$,

$$f'(x) = -\tfrac{1}{2}(a - x)^{-\frac{1}{2}},$$

$$F'(x) = \frac{\pi}{2}\cos\left\{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\right\}.\ \frac{-a}{(a+x)^{\frac{3}{2}}.\,(a-x)^{\frac{1}{2}}} = \frac{-\pi a}{2\,(2a)^{\frac{3}{2}}\,(a-x)^{\frac{1}{2}}},$$

and consequently

$$\frac{f(x)}{F(x)} = \frac{f'(x)}{F'(x)} = \frac{(2a)^{\frac{3}{2}}}{\pi a},$$

and $$\phi(x) = (2a)^{\frac{1}{2}}.\ \frac{f(x)}{F(x)} = \frac{4a}{\pi}.$$

The value of $\phi(x)$ may be obtained also, and in fact more simply, without the aid of differentials: thus

$$\phi(x) = \frac{2}{\pi}(a+x)\cos\left\{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\right\}.\ \frac{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}}{\sin\left\{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\right\}}:$$

but, when $x = a$,

$$\frac{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}}{\sin\left\{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\right\}} = \frac{\theta}{\sin\theta},\ \text{when } \theta = 0,$$

$$= 1:$$

hence $$\phi(a) = \frac{4a}{\pi}.$$

Ex. 5. To evaluate

$$\phi(x) = \frac{\sin \pi x - \pi x \cos \pi x}{2x^2 \sin \pi x},\ \text{when } x = 0.$$

Here $f(x) = \sin \pi x - \pi x \cos \pi x = 0,$

$F(x) = 2x^2 \sin \pi x = 0:$

$f'(x) = \pi \cos \pi x - \pi \cos \pi x + \pi^2 x \sin \pi x = \pi^2 x \sin \pi x,$

$F'(x) = 4x \sin \pi x + 2\pi x^2 \cos \pi x.$

Hence, when $x = 0$,

$$\phi(x) = \frac{f'(x)}{F'(x)} = \frac{\pi^2 \sin \pi x}{4 \sin \pi x + 2\pi x \cos \pi x} = \frac{0}{0}.$$

Put $f_1(x) = \pi^2 \sin \pi x,$

$F_1(x) = 4 \sin \pi x + 2\pi x \cos \pi x :$

then $f_1'(x) = \pi^3 \cos \pi x = \pi^3,$

$F_1'(x) = 4\pi \cos \pi x + 2\pi \cos \pi x - 2\pi^2 x \sin \pi x = 6\pi.$

Hence $$\phi(0) = \frac{\pi^3}{6\pi} = \frac{\pi^2}{6}.$$

Evaluation of Functions of the form $\frac{\infty}{\infty}$.

64. The rule for the evaluation of functions of x, which for particular values of the variable assume the form $\frac{0}{0}$, is applicable also for the evaluation of functions which assume the form $\frac{\infty}{\infty}$.

Let $$\phi(x) = \frac{f(x)}{F(x)} = \frac{\infty}{\infty}, \quad \text{when } x = x_0:$$

then $$\phi(x) = \frac{\frac{1}{F(x)}}{\frac{1}{f(x)}} = \frac{0}{0}, \quad \text{when } x = x_0.$$

Let $$F_1(x) = \frac{1}{F(x)}, \quad f_1(x) = \frac{1}{f(x)}:$$

then $$\phi(x) = \frac{F_1(x)}{f_1(x)} = \frac{0}{0}, \quad \text{when } x = x_0.$$

Hence, by the theorem of Art. (63), for this value of x,

$$\phi(x) = \frac{F_1'(x)}{f_1'(x)};$$

but $$F_1'(x) = -\frac{F'(x)}{\{F(x)\}^2}, \quad f_1'(x) = -\frac{f'(x)}{\{f(x)\}^2}:$$

hence, when $x = x_0$,

$$\phi(x) = \frac{F'(x)}{f'(x)} \cdot \left\{\frac{f(x)}{F(x)}\right\}^2 = \frac{F'(x)}{f'(x)} \cdot \{\phi(x)\}^2,$$

or $$\phi(x) = \frac{f'(x)}{F'(x)},$$

which establishes the proposition.

If $f'(x_0) = \infty$ and $F'(x_0) = \infty$, then, by the application of the same principle, it is plain that, for this particular value of x,

$$\frac{f(x)}{F(x)} = \frac{f'(x)}{F'(x)} = \frac{f''(x)}{F''(x)},$$

and so on indefinitely.

Ex. 1. To find the value of

$$\phi(x) = e^{-\frac{1}{x}}.(1 - \log x), \quad \text{when } x = +0,$$

where $+0$ denotes the limit of positive magnitude.

Under a different form,

$$\phi(x) = \frac{1 - \log x}{e^{\frac{1}{x}}} = \frac{\infty}{\infty}:$$

$$f(x) = 1 - \log x,$$

$$F(x) = e^{\frac{1}{x}}:$$

$$f'(x) = -\frac{1}{x},$$

$$F'(x) = -\frac{1}{x^2}e^{\frac{1}{x}}:$$

hence $$\phi(x) = \frac{f'(x)}{F'(x)} = \frac{x}{e^{\frac{1}{x}}} = \frac{0}{\infty} = 0.$$

Ex. 2. To find the value of

$$\phi(x) = \frac{(\log x)^2}{x^2}, \quad \text{when } x = \infty:$$

$$f(x) = (\log x)^2 = \infty,$$

$$F(x) = x^2 = \infty:$$

$$f'(x) = \frac{2}{x}\log x,$$

$$F'(x) = 2x:$$

hence $$\phi(x) = \frac{f'(x)}{F'(x)} = \frac{\log x}{x^2} = \frac{\infty}{\infty}.$$

Put
$$f_1(x) = \log x = \infty,$$
$$F_1(x) = x^2 = \infty:$$

then
$$f_1'(x) = \frac{1}{x} = 0,$$
$$F_1'(x) = 2x = \infty;$$

and therefore
$$\phi(x) = \frac{f_1(x)}{F_1(x)} = \frac{f_1'(x)}{F_1'(x)} = \frac{0}{\infty} = 0.$$

It would be more easy, in order to evaluate $\phi(x)$, first to find the value of
$$\Psi(x) = \frac{\log x}{x} = \frac{\infty}{\infty},$$
and then to square the result. Putting
$$f(x) = \log x,$$
$$F(x) = x;$$

we have
$$f'(x) = \frac{1}{x} = 0,$$
$$F'(x) = 1:$$

hence
$$\Psi(\infty) = \frac{f'(\infty)}{F'(\infty)} = 0,$$

and thus
$$\phi(x) = \{\Psi(x)\}^2 = 0, \qquad \text{when } x = \infty.$$

Ex. 3. To find the value of
$$\phi(x) = \frac{x^m}{a^{x^n}}, \qquad \text{when } x = \infty;$$
a, m, n, being positive quantities, and a being greater than unity.

If we differentiate the expression
$$\frac{x^m}{a^{x^n}}$$
m times above and below, the series of resulting fractions will always be of the form $\frac{\infty}{\infty}$, up to the m^{th} differentiation, which will give a fraction
$$\frac{C}{\Psi(x)},$$

C being finite and $\Psi(x)$ infinite, when $x = \infty$: thus we see that

$$\phi(\infty) = 0.$$

Failure of the method of Differentials for the Evaluation of Indeterminate Functions.

65. It occasionally happens, when, for a particular value of x,

$$\phi(x) = \frac{f(x)}{F(x)} = \frac{0}{0}, \quad \text{or} \ = \frac{\infty}{\infty},$$

that the application of the above rules is inadequate for the evaluation of $\phi(x)$. By virtue of these rules we may take $\frac{f'(x)}{F'(x)}$ as an equivalent for $\frac{f(x)}{F(x)}$; we may modify $\frac{f'(x)}{F'(x)}$ as we please by cancelling or introducing common factors above and below; and, supposing $\frac{f_1(x)}{F_1(x)}$ to represent the modified fraction, we are at liberty, provided that

$$\frac{f_1(x)}{F_1(x)} = \frac{0}{0}, \quad \text{or} \ = \frac{\infty}{\infty},$$

to put

$$\phi(x) = \frac{f_1'(x)}{F_1'(x)},$$

and so on indefinitely, until we at length arrive at a fraction of which at least both the numerator and denominator are not simultaneously zero or simultaneously infinite. Sometimes, however, we are unable by the application of these combined operations to rescue the function from its nugatory form, the indeterminateness perpetually presenting itself again and again in the successive functions either in the shape of $\frac{0}{0}$ or of $\frac{\infty}{\infty}$.

Ex. Suppose that

$$\phi(x) = \frac{f(x)}{F(x)} = \frac{a^{x^m}}{b^{x^n}};$$

and that $x = \infty$: and that it is required to find the value of $\phi(x)$ for this value of x, m and n being positive numbers, and both a and b being greater than unity.

$$f'(x) = mx^{m-1} . \log a . a^{x^m},$$

$$F'(x) = nx^{n-1} . \log b . b^{x^n}:$$

hence $$\phi(x) = \frac{f'(x)}{F'(x)} = \frac{m \log a}{n \log b} \cdot \frac{x^m . a^{x^m}}{x^n . b^{x^n}}$$

$$= \frac{m \log a}{n \log b} \cdot \frac{a^{x^m}}{\dfrac{b^{x^n}}{x^{m-n}}}, \qquad \text{if } m \text{ be} > n,$$

$$= \frac{\infty}{\infty}, \text{ by Art. (64), Ex. 3.}$$

We shall fall into the same difficulty at each succeeding operation. In fact, the method of differentiation must not be considered as a universal rule for evaluating indeterminate functions, but merely as an instrument frequently of great use for this purpose.

Evaluation of Indeterminate Functions of several Independent Variables.

66. Suppose that, to take the case of two independent variables,

$$\phi(x, y) = \frac{f(x, y)}{F(x, y)} = \frac{0}{0},$$

when x, y, receive respectively particular values x_0, y_0; the variables x and y being subject to no equation of connection.

Generally

$$\phi(x + \delta x,\ y + \delta y) = \frac{f(x, y) + \Delta F(x, y)}{F(x, y) + \Delta F(x, y)},$$

$\Delta f(x, y)$ and $\Delta F(x, y)$ denoting the total differences of $f(x, y)$ and $F(x, y)$.

Suppose that $x = x_0$, $y = y_0$: then

$$\phi(x_0 + \delta x_0,\ y_0 + \delta y_0) = \frac{\Delta f(x, y)}{\Delta F(x, y)},$$

supposing that, in the expressions for $\Delta f(x, y)$, $\Delta F(x, y)$, which are certain functions of x, y, δx, δy, we finally substitute

$x_0, y_0, \delta x_0, \delta y_0$, for $x, y, \delta x, \delta y$. Proceeding to the limit we have

$$\phi(x_0, y_0) = \frac{Df(x, y)}{DF(x, y)},$$

x_0, y_0, dx_0, dy_0, being finally substituted for x, y, dx, dy.

But
$$Df(x, y) = \frac{df}{dx}\,dx + \frac{df}{dy}\,dy,$$

$$DF(x, y) = \frac{dF}{dx}\,dx + \frac{dF}{dy}\,dy:$$

hence
$$\phi(x_0, y_0) = \frac{\dfrac{df}{dx}\,dx + \dfrac{df}{dy}\,dy}{\dfrac{dF}{dx}\,dx + \dfrac{dF}{dy}\,dy}.$$

Since the ratio of dx to dy is undefined, it appears that the value of $\phi(x_0, y_0)$ is generally indeterminate: suppose however that, when $x = x_0$ and $y = y_0$, either

$$\frac{df}{dx} = 0, \quad \text{and} \quad \frac{dF}{dx} = 0,$$

or
$$\frac{df}{dy} = 0, \quad \text{and} \quad \frac{dF}{dy} = 0.$$

In the former case
$$\phi(x_0, y_0) = \frac{\dfrac{df}{dy}}{\dfrac{dF}{dy}},$$

in the latter,
$$\phi(x_0, y_0) = \frac{\dfrac{df}{dx}}{\dfrac{dF}{dx}}:$$

in both these cases the indeterminateness disappears.

If the four quantities

$$\frac{df}{dx}, \quad \frac{dF}{dx}, \quad \frac{df}{dy}, \quad \frac{dF}{dy},$$

be simultaneously zero, we must proceed to second differentials, when

$$\phi(x_0, y_0) = \frac{D^2 f(x, y)}{D^2 F(x, y)} = \frac{\dfrac{d^2 f}{dx^2} dx^2 + 2\,\dfrac{d^2 f}{dx\,dy}\,dx\,dy + \dfrac{d^2 f}{dy^2}\,dy^2}{\dfrac{d^2 F}{dx^2} dx^2 + 2\,\dfrac{d^2 F}{dx\,dy}\,dx\,dy + \dfrac{d^2 F}{dy^2}\,dy^2},$$

an expression generally indeterminate by reason of the indefiniteness of the ratio of dx to dy. If all the partial differentials of f and F of the second order are zero, we must proceed to the third order, and so on.

The extension of the preceding considerations to indeterminate functions of any number of independent variables is obvious. We have considered only the case of indetermination of the form $\frac{0}{0}$: the application of the method, however, to that of the form $\frac{\infty}{\infty}$ may be established just as in the instance of a single independent variable.

Ex. 1. Let

$$\phi(x, y) = \frac{\log x + \log y}{x + 2y - 3}; \qquad x_0 = 1, \ y_0 = 1.$$

Here

$$\frac{df}{dx} = \frac{1}{x} = 1, \qquad \frac{df}{dy} = \frac{1}{y} = 1,$$

$$\frac{dF}{dx} = 1, \qquad \frac{dF}{dy} = 2:$$

whence

$$\phi(x_0, y_0) = \frac{dx + dy}{dx + 2dy} = \frac{1 + a}{1 + 2a},$$

where a is an arbitrary quantity. Thus $\phi(x_0, y_0)$ may have any value from $-\infty$ to $+\infty$.

Ex. 2. Let

$$\phi(x, y) = \frac{(x - 1)^{\frac{3}{2}} + y^{\frac{3}{2}} - 1}{(x^2 - 1)^{\frac{3}{2}} - y + 1}; \qquad x_0 = 1, \ y_0 = 1.$$

Here
$$\frac{df}{dx} = \tfrac{3}{2}(x-1)^{\frac{1}{2}} = 0, \qquad \frac{df}{dy} = \tfrac{3}{2}y^{\frac{1}{2}} = \tfrac{3}{2},$$
$$\frac{dF}{dx} = 3x(x^2-1)^{\frac{1}{2}} = 0, \quad \frac{dF}{dy} = -1:$$

hence $\phi(x_0, y_0) = -\frac{3}{2}$, a determinate value.

Ex. 3. Let $\phi(x, y) = \dfrac{(x+y)^2}{x^2+y^2}$, $\quad x_0 = 0,\ y_0 = 0.$

Then
$$\frac{df}{dx} = 2(x+y) = 0, \quad \frac{df}{dy} = 2(x+y) = 0,$$
$$\frac{dF}{dx} = 2x = 0, \qquad \frac{dF}{dy} = 2y = 0.$$

The partial differentials of the first order being zero, we must proceed to differentials of the second order.

$$\frac{d^2f}{dx^2} = 2, \qquad \frac{d^2f}{dx\,dy} = 2, \qquad \frac{d^2f}{dy^2} = 2,$$
$$\frac{d^2F}{dx^2} = 2, \qquad \frac{d^2F}{dx\,dy} = 0, \qquad \frac{d^2F}{dy^2} = 2.$$

Hence
$$\phi(x_0, y_0) = \frac{2\,dx^2 + 4\,dx\,dy + 2\,dy^2}{2\,dx^2 + 2\,dy^2}$$
$$= \frac{(1+a)^2}{1+a^2} = 1 + \frac{2}{a + \frac{1}{a}},$$

a being an arbitrary quantity. The value of $\phi(x_0, y_0)$ is therefore indeterminate, within certain limits; its greatest and least values corresponding to the least positive and least negative values of $a + \dfrac{1}{a}$. Suppose that

$$a + \frac{1}{a} = \beta,$$

then
$$4a^2 - 4\beta a + \beta^2 = \beta^2 - 4,$$
$$2a - \beta = \pm(\beta^2 - 4)^{\frac{1}{2}}:$$

hence $+2$ and -2 are the least positive and negative values of

β or $\alpha + \frac{1}{\alpha}$. It appears therefore that $\phi(x_0, y_0)$ may have any value whatever from 0 to + 2.

Evaluation of indeterminate implicit Functions of a single Variable.

67. Suppose that an equation

$$f(x, y) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

is satisfied identically by a certain value x_0 of x, whatever be the value of y. The function y will for this value of x appear to be indeterminate.

Differentiating the proposed equation, we get

$$\frac{df}{dx}\,dx + \frac{df}{dy}\,dy = 0 \ldots\ldots\ldots\ldots (2).$$

But since, when $x = x_0$, $f(x, y)$ has a constant value zero for all values of y whatever, it follows that in this case also

$$\frac{df}{dy} = 0:$$

hence we have, when $x = x_0$,

$$\frac{df}{dx}\,dx = 0, \quad \text{or} \quad \frac{df}{dx} = 0 \ldots\ldots\ldots\ldots (3).$$

The value of y_0 must be determined from the equation (3). In case the equation (3) be satisfied identically for all values of y, we must, the function $\frac{df}{dx}$ now occupying the place of the original function f, proceed to determine y_0 from the equation

$$\frac{d^2f}{dx^2} = 0,$$

and so on, until the indeterminateness is eradicated.

Ex. 1. Suppose that

$$f(x, y) = mx^2 - x + \log(1 + xy) = 0, \quad x_0 = 0.$$

Then $$\frac{df}{dx} = 2mx - 1 + \frac{y}{1 + xy} = 0,$$

whence $$-1 + \frac{y_0}{1} = 0, \quad \text{or } y_0 = 1.$$

Ex. 2. Let

$$f(x, y) = (y^2 - 1)\,x^2 - y\,\{\log(1 + x)\}^2 = 0, \quad x_0 = 0.$$

Here the equation $$\frac{df}{dx} = 0$$

gives us, for the determination of y_0,

$$2(y^2 - 1)\,x - \frac{2y \log(1 + x)}{1 + x} = 0,$$

or $$F(x, y) = (y^2 - 1)(x^2 + x) - y \log(1 + x) = 0:$$

but this equation is identically satisfied by $x = 0$: we must therefore differentiate again with regard to x.

The equation $$\frac{dF}{dx} = 0$$

gives us, for finding y_0,

$$(y^2 - 1)(2x + 1) - \frac{y}{1 + x} = 0,$$

whence $$y_0^2 - 1 - y_0 = 0,$$

a quadratic giving two values $\dfrac{\pm\sqrt{5} + 1}{2}$ for y_0.

CHAPTER VI.

MAXIMA AND MINIMA.

Definition of a Maximum and Minimum.

68. Let $y = f(x)$, and suppose that, as x gradually increases, through a particular value x_0, from a value $x_0 - h$ to a value $x_0 + h$, h being an indefinitely small positive quantity, y increases, as x increases from $x_0 - h$ to x_0, and decreases, as x increases from x_0 to $x_0 + h$. In this case y is said to have a *maximum* value when $x = x_0$. If y decreases, as x increases from $x_0 - h$ to x_0, and increases, as x increases from x_0 to $x_0 + h$, then y is said to have a *minimum* value when $x = x_0$. The words *increase* and *decrease* are here used in algebraical senses to indicate progress from $-\infty$ towards $+\infty$, and from $+\infty$ towards $-\infty$, respectively.

Preliminary Lemma.

69. Before proceeding to investigate a rule for determining the maximum or minimum values of y, it will be necessary to premise the following lemma.

Lemma. If u be a function of x; then, accordingly as u is increasing or decreasing as x increases, $\frac{du}{dx}$ will be respectively positive or negative, and conversely.

Suppose that, when x increases to a value $x + \delta x$, δx being very small, u becomes $u + \delta u$; then, if u be increasing with the increase of x, δu must be positive, and if u be decreasing, δu must be negative: hence $\frac{\delta u}{\delta x}$ must be positive in the former and negative in the latter case; and this must be true ultimately when δx is diminished without limit. Hence $\frac{du}{dx}$ must be positive in the former and negative in the latter case.

Conversely, since $\frac{du}{dx}$ is the limiting value of $\frac{\delta u}{\delta x}$, it is evident that, accordingly as $\frac{du}{dx}$ is positive or negative, $\frac{\delta u}{\delta x}$ must also be positive or negative when δx is sufficiently small, and that consequently u must be increasing or decreasing with the increase of x accordingly as $\frac{du}{dx}$ is positive or negative.

In other words, that u may be increasing or decreasing with an increase of x, it is sufficient and necessary that $\frac{du}{dx}$ be positive in the former and negative in the latter case.

Rule for finding Maxima and Minima.

70. By the definition of a maximum or minimum given above, and by virtue of the Lemma of the preceding article, we see that, for a *maximum* value of y, it is sufficient and necessary that $\frac{dy}{dx}$ change sign from + to − as x passes from $x_0 - h$ to $x_0 + h$, being positive for the former range of values of x and negative for the latter. From these sufficient and necessary conditions it follows that, when $x = x_0$, $\frac{dy}{dx}$ must become either zero or infinity, since a function of x can change sign only in passing through one or other of these values. Similarly, that $x = x_0$ may correspond to a *minimum* value of y, it appears that $\frac{dy}{dx}$ must pass from − to + as x passes from $x_0 - h$ to $x_0 + h$ and become, as in the case of a *maximum*, either zero or infinity when $x = x_0$.

We may now enunciate a general rule for finding the maximum and minimum values of y.

RULE. Obtain all the values of x which satisfy either of the two equations

$$f'(x) = 0, \qquad f'(x) = \infty:$$

if any one of these values of x be such that, as x increases through it, $f'(x)$ changes sign from + to −, it will correspond to a maximum value of y; if it be such that, as x increases through it, $f'(x)$ changes sign from − to +, it will correspond to a minimum value of y; and if it be such that, as x increases through it, $f'(x)$ does not change sign, it will correspond neither to a maximum nor to a minimum value of y.

Ex. 1. To find whether, n being a positive integer,

$$y = (x - a)^n$$

has a maximum or minimum value.

Here $$\frac{dy}{dx} = n(x - a)^{n-1}:$$

equating to zero the value of $\frac{dy}{dx}$, we have

$$x - a = 0, \qquad x = a.$$

Putting in the expression for $\frac{dy}{dx}$ first $a - h$ and next $a + h$, we see that, if n be even, $\frac{dy}{dx}$ will be negative in the former case and positive in the latter; and that, if n be odd, $\frac{dy}{dx}$ will have the same sign in both cases. Hence, if n be even, $x = a$ gives zero as a minimum value of y, and, if n be odd, y has neither a maximum nor a minimum value.

Ex. 2. Let $$y = x^2 + 3x + 2:$$

then $$\frac{dy}{dx} = 2x + 3 = 0, \qquad x = -\tfrac{3}{2}.$$

If $x = -\frac{3}{2} - h$, it is evident that $\frac{dy}{dx}$ is negative, and positive when $x = -\frac{3}{2} + h$. Hence $x = -\frac{3}{2}$ renders y a minimum, its value being

$$(-\tfrac{3}{2})^2 + 3(-\tfrac{3}{2}) + 2 = -\tfrac{1}{4}.$$

Ex. 3. Let
$$y = \frac{(x+3)^4}{(x+2)^6}:$$

then
$$\frac{dy}{dx} = -\frac{2(x+3)^3(x+5)}{(x+2)^7}.$$

Putting $\frac{dy}{dx} = 0$, we have
$$x = -3, \qquad \text{or } x = -5:$$
putting $\frac{dy}{dx} = \infty$, we have $x = -2$.

Now
$$\frac{dy}{dx} = \pm, \quad \text{if } x = -2 \mp h;$$
$$\frac{dy}{dx} = \mp, \quad \text{if } x = -3 \mp h;$$
and
$$\frac{dy}{dx} = \pm, \quad \text{if } x = -5 \mp h.$$

Hence, if $x = -2$, $y = \infty$, a maximum,
if $x = -3$, $y = 0$, a minimum,
and if $x = -5$, $y = \frac{16}{729}$, a maximum.

Abbreviation of Operation.

71. Suppose $\phi(x)$ to be any function of x which for all possible values of x has the same sign as $f'(x)$. Then it is evident that in the rule which we have enunciated for finding the maximum and minimum values of $f(x)$, we may replace $f'(x)$ by $\phi(x)$. This change will frequently abbreviate the processes of investigation. Thus if, for instance,
$$f'(x) = \Psi(x) \, . \, \phi(x),$$
where $\Psi(x)$ is a function of x essentially positive, we may reject $\Psi(x)$ and take $\phi(x)$ in place of $f'(x)$.

Ex. 1. To find the maximum or minimum values of y, when
$$y = \frac{x-1}{x(x+1)}.$$

In this case

$$\frac{dy}{dx}=\frac{1+2x-x^2}{x^2(x+1)^2}=\Psi(x).(1+2x-x^2):$$

now $\Psi(x)$ cannot change sign: put therefore

$$\phi(x)=1+2x-x^2=0:$$

we have then $x=1-\sqrt{2}$, or $x=1+\sqrt{2}$.

It is easily seen that

$$\phi(x)=\mp, \text{ if } x=1-\sqrt{2}\mp h,$$

and $$\phi(x)=\pm, \text{ if } x=1+\sqrt{2}\mp h:$$

hence $x=1-\sqrt{2}$ gives a minimum and $x=1+\sqrt{2}$ gives a maximum value of y.

By the rejection of the factor

$$\Psi(x)=\frac{1}{x^2(x+1)^2},$$

it will be observed that we escape the trouble of examining the consequences of putting $f'(x)=\infty$.

Ex. 2. Let $$y=\frac{(x-1)^2}{(x+1)^3}.$$

Then $$\frac{dy}{dx} \text{ or } f'(x)=\frac{(x-1)(5-x)}{(x+1)^4}=\Psi(x).\phi(x),$$

where $$\phi(x)=(x-1)(5-x).$$

Putting $\phi(x)=0$, we see that

$$x=1 \quad\text{or}\quad x=5.$$

Also $$\phi(x)=\mp, \text{ if } x=1\mp h,$$

and $$\phi(x)=\pm, \text{ if } x=5\mp h.$$

Hence, if $x=1$, $y=0$, a minimum,

and if $x=5$, $y=\frac{2}{27}$, a maximum.

Alternation of Maxima and Minima.

72. Supposing y to be a function $f(x)$ of x, which has several maxima and minima, then, as x keeps continuously

increasing, the maximum and minimum values of y will occur alternately. This will easily be seen when we consider that whenever the sign of $f'(x)$ changes from $+$ to $-$, y is a maximum, and, whenever it changes from $-$ to $+$, a minimum; and that a change from $+$ to $-$ can be succeeded only by a change from $-$ to $+$, and conversely.

Modified method of finding Maxima and Minima.

73. Suppose that $f(x)$ has a maximum value when $x = x_0$, and that none of the derived functions

$$f'(x),\ \ f''(x),\ \ f'''(x),\ \ldots\ldots$$

become infinite when $x = x_0$. Then, since $f'(x)$ decreases from $+$, through 0, to $-$, as x passes from $x_0 - h$, through x_0, to $x_0 + h$, it appears, by the Lemma of Art. (69), that $f''(x)$ must have the sign $-$ for this range of values of x. If $f''(x) = -0$, when $x = x_0$, the symbol -0 being used to denote zero regarded as a limiting state of negative magnitude, then, when $x = x_0$, it is evident that $f''(x)$ has a maximum value: from this it follows that, since $f''(x)$ now occupies the place of $f(x)$, $f'''(x)$ will change sign from $+$, through 0, to $-$, and $f^{iv}(x)$ will have the sign $-$, as x ranges from $x_0 - h$ to $x_0 + h$. If $f^{iv}(x) = -0$, when $x = x_0$, then, $f^{iv}(x)$ now occupying the place of $f''(x)$, we see that $f^{v}(x)$ will change sign from $+$, through 0, to $-$, and $f^{vi}(x)$ will have the sign $-$, as x ranges from $x_0 - h$ to $x_0 + h$. We may proceed with this reasoning from step to step until we arrive at a derived function of an even order which does not vanish when $x = x_0$. Our final conclusion is evidently that, for a maximum value of $f(x)$, we must have $f'(x) = 0$, and that, of the differential coefficients of $f(x)$, the first which, for a corresponding value of x, does not vanish, must be of an even order, and must be negative.

By precisely the same form of reasoning, *mutatis mutandis*, we may see that, for a minimum value of $f(x)$, the sufficient and necessary conditions are that

$$f'(x) = 0,$$

and that of the derived functions $f''(x)$, $f'''(x)$, ... the first which, for a corresponding value of x, does not vanish, shall be of an even order and shall be positive.

Hence, to find the maxima and minima of a function $f(x)$, we must equate its first differential coefficient to zero, and thence obtain corresponding values of x: we must then keep differentiating the function until, for each of these values of x, we arrive at a differential coefficient which does not vanish: if, for any one of these values of x, this final differential coefficient is of an even order, the corresponding value of $f(x)$ will be a maximum or a minimum accordingly as the final differential coefficient is negative or positive. If the final differential coefficient is of an odd order, the corresponding value of $f(x)$ will be neither a maximum nor a minimum.

Ex. 1. Let $$y = \frac{x}{x^2+1}:$$

then $$\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2},$$

or $$f'(x) = \Psi(x).(1-x^2),$$

$\Psi(x)$ being an essentially positive factor. Take then, instead of $f'(x)$, $$\phi(x) = 1 - x^2 = 0:$$

then $$x = 1, \quad \text{or} \quad x = -1:$$

$$\text{if } x = 1, \qquad \phi'(x) = -2x = -;$$

$$\text{if } x = -1, \qquad \phi'(x) = -2x = +.$$

Hence $x = +1$ makes y a maximum, and $x = -1$ makes it a minimum.

Ex. 2. Let $y = x^4 - 8x^3 + 22x^2 - 24x + 12$.

Then $$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0:$$

the roots of this equation are 1, 2, 3. Now

$$\frac{d^2y}{dx^2} = 12x^2 - 48x + 44:$$

hence
$$\frac{d^2y}{dx^2} = +, \quad \text{when } x = 1,$$

$$\frac{d^2y}{dx^2} = -, \quad \text{when } x = 2,$$

$$\frac{d^2y}{dx^2} = +, \quad \text{when } x = 3.$$

Thus we see that for the values 1, 2, 3, of x, y is respectively a minimum, a maximum, a minimum.

Ex. 3. Let
$$y = e^x + 2\cos x + e^{-x}:$$
then, when $x = 0$,
$$\frac{dy}{dx} = e^x - 2\sin x - e^{-x} = 0,$$

$$\frac{d^2y}{dx^2} = e^x - 2\cos x + e^{-x} = 0,$$

$$\frac{d^3y}{dx^3} = e^x + 2\sin x - e^{-x} = 0,$$

$$\frac{d^4y}{dx^4} = e^x + 2\cos x + e^{-x} = 4.$$

Hence $x = 0$ gives for y a minimum value 4.

Ex. 4. Let
$$y = b + c(x - a)^{\frac{2}{3}}:$$
then
$$\frac{dy}{dx} = \tfrac{2}{3}c(x - a)^{-\frac{1}{3}} = c(x - a) \,.\, \Psi(x),$$
where $\Psi(x) = \frac{2}{3}(x - a)^{-\frac{4}{3}}$, a quantity essentially positive.

Instead, therefore, of $\dfrac{dy}{dx}$ or $f'(x)$, we may take
$$\phi(x) = c(x - a) = 0;$$
whence we get $x = a$; also
$$\phi'(x) = c.$$
If therefore c be a positive quantity, $x = a$ makes y a minimum; if c be a negative quantity, $x = a$ makes y a maximum.

Ex. 5. Let
$$y = \frac{(x+3)^3}{(x+2)^2}.$$

Then
$$\frac{dy}{dx} = \frac{x(x+3)^2}{(x+2)^3}$$
$$= x(x+2) \,.\, \Psi(x),$$

where
$$\Psi(x) = \frac{(x+3)^2}{(x+2)^4},$$

an essentially positive quantity.

Hence, instead of $\frac{dy}{dx}$ or $f'(x)$, take
$$\phi(x) = x(x+2) = 0:$$

then
$$x = 0, \quad \text{or} \quad x = -2;$$

$$\text{if } x = 0, \quad \phi'(x) = x + 2 = +, \quad y = \tfrac{27}{4}, \quad \text{a minimum:}$$
$$\text{if } x = -2, \quad \phi'(x) = x = -, \quad y = \infty, \quad \text{a maximum.}$$

From this and the preceding example it appears that, although either $f(x)$, or its derived functions of sufficiently high orders, may become infinite for a value of x which makes $f(x)$ a maximum or a minimum, yet, if we replace $f'(x)$ by an appropriate function $\phi(x)$ which has always the same sign as $f'(x)$, we may often apply with advantage the rule of Art. (73) for finding such a value of x.

Abbreviation of Operation.

74. Suppose that, for a certain function $f(x) = y$,
$$\frac{dy}{dx} = uv,$$

u being a factor which vanishes when $x = a$, while v remains finite. Then, if $\frac{d^r u}{dx^r}$ be the first differential coefficient of u which does not vanish when $x = a$, it is easy to see that
$$\frac{d^r y}{dx^r} = v\,\frac{d^r u}{dx^r}, \quad \text{when } x = a,$$

and that all the differential coefficients of y of lower orders than the r^{th} will be equal to zero. This consideration enables us to ascertain whether a particular value a of x, which makes $\frac{dy}{dx} = 0$, corresponds to a maximum or a minimum value of y, without being driven to the necessity of obtaining the general expression for the value of $\frac{d^ry}{dx^r}$.

Ex. 1. Suppose that y is such a function of x that

$$\frac{dy}{dx} = (x-1)(x-2)(x-3)(x-4),$$

and suppose that we desire to know whether $x = 1$, which makes $\frac{dy}{dx} = 0$, corresponds to a maximum or to a minimum value of y. We have, if $x = 1$,

$$\frac{d^2y}{dx^2} = (x-2)(x-3)(x-4) = (-)(-)(-) = -:$$

which shews that $x = 1$ corresponds to a maximum value of y.

Ex. 2. Suppose that

$$\frac{dy}{dx} = (x-1)^3(x-2)(x-3)(x-4);$$

then $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = 0$, if $x = 1$, and

$$\frac{d^4y}{dx^4} = 1.2.3\,(x-2)(x-3)(x-4) = -:$$

which shews that $x = 1$ corresponds to a maximum value of y.

Maxima and Minima of implicit Functions of a single Variable.

75. In the preceding articles we have investigated the method of finding the maxima and minima of an explicit function of a single variable. We proceed now to the consideration of those cases in which the function is involved implicitly with its variable. Let y be a function of x by virtue of an equation

$$u = f(x, y) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating this equation, we get

$$\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

From (2), since $\frac{dy}{dx} = 0$ or $= \infty$, is an essential condition for a maximum or minimum value of y, we have

$$-\frac{\frac{du}{dx}}{\frac{du}{dy}} = 0 \text{ or } = \infty \ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we may obtain those systems of values of x and y which alone can correspond to maximum or minimum values of y. In order to find whether a value x_0 of x so discovered does really give a maximum or minimum value of y, we must substitute, in the expression

$$-\frac{\frac{du}{dx}}{\frac{du}{dy}},$$

$x_0 - h$, $x_0 + h$, successively for x, and the corresponding values of y, and see whether the expression changes sign from $\pm$ to $\mp$, the additional conditions for a maximum and a minimum value respectively.

For those values of x which do not render $\frac{dy}{dx}$, or any of the higher order of differential coefficients of y, equal to infinity, we may proceed to find, by implicit differentiation, the values of $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ until we arrive at one which does not vanish for a value of x which makes $\frac{dy}{dx} = 0$. If the order of this final differential coefficient be even, the value of x gives a maximum or a minimum value of y, accordingly as the sign of the differential coefficient is negative or positive.

Ex. Let $$x^3 - 3axy + y^3 = 0 \ldots\ldots\ldots\ldots\ldots (1)':$$

then $$x^2 - ay + (y^2 - ax)\frac{dy}{dx} = 0 \,\ldots\ldots\ldots\ldots (2)'.$$

Supposing that $\frac{dy}{dx} = 0$, we have

$$x^2 - ay = 0 \,\ldots\ldots\ldots\ldots\ldots\ldots (3)'.$$

Eliminating y between $(1)'$ and $(3)'$, we get

$$x^6 - 2a^3x^3 = 0,$$

whence $$x = 0, \quad \text{or} \quad x = a\sqrt[3]{2},$$

and therefore respectively

$$y = 0, \quad \text{or} \quad y = a\sqrt[3]{4}.$$

Differentiating $(2)'$ we have, taking the values $a\sqrt[3]{2}$, $a\sqrt[3]{4}$, for x, y,

$$2x + (y^2 - ax)\frac{d^2y}{dx^2} = 0,$$

$$2a\sqrt[3]{2} + a^2 . a\sqrt[3]{2} . \frac{d^2y}{dx^2} = 0,$$

$$\frac{d^2y}{dx^2} = -\frac{2}{a}, \text{ a negative quantity:}$$

hence $a\sqrt[3]{4}$ is a maximum value of y.

If $x = 0$ and $y = 0$, then $\frac{dy}{dx}$ derived from $(2)'$ takes the indeterminate form $\frac{0}{0}$: we may however extricate ourselves from this difficulty by differentiating $(2)'$ successively until we obtain an equation in which $\frac{dy}{dx}$ no longer presents itself under this form. Thus, after one differentiation,

$$(y^2 - ax)\frac{d^2y}{dx^2} + 2y\frac{dy^2}{dx^2} - 2a\frac{dy}{dx} + 2x = 0,$$

and, after a second differentiation,

$$(y^2 - ax)\frac{d^3y}{dx^3} + \left(6y\frac{dy}{dx} - 3a\right)\frac{d^2y}{dx^2} + 2\frac{dy^3}{dx^3} + 2 = 0.$$

When $x = 0$ and $y = 0$, this equation becomes

$$-3a\frac{d^2y}{dx^2} + 2 = 0,$$

whence $$\frac{d^2y}{dx^2} = \frac{2}{3a}, \text{ a positive quantity.}$$

Thus we see that the system of values $x = 0$, $y = 0$, corresponds to a minimum value of y.

We will discuss this example also by examining directly whether $\frac{dy}{dx}$ changes sign as x passes through 0 and $a\sqrt[3]{2}$.

First, taking the value 0 of x, put h for x in (1)′, h being very small: then

$$h^3 - 3ahy + y^3 = 0,$$

or approximately, $$- 3ahy + y^3 = 0,$$

whence $$y = 0, \quad \text{or } y = \pm (3a)^{\frac{1}{2}} . h^{\frac{1}{2}}.$$

Hence, if $y = 0$, we see that,

when $$x = -h, \quad \frac{dy}{dx} = -\frac{h^2}{+ah} = -,$$

when $$x = +h, \quad \frac{dy}{dx} = -\frac{h^2}{-ah} = +:$$

hence $x = 0$ gives $y = 0$ as a minimum value of y.
The values $\pm (3a)^{\frac{1}{2}} . h^{\frac{1}{2}}$ of y must be rejected because they are impossible when $-h$ is put for $+h$.

We will next take the system

$$x = a\sqrt[3]{2}, \quad a\sqrt[3]{4}.$$

Putting $x = a\sqrt[3]{2} + h$, $y = a\sqrt[3]{4} + k$, in (1)′, we have nearly

$$2a^3 + 3a^2 (2)^{\frac{2}{3}} . h - 3a \left\{a (2)^{\frac{1}{3}} + h\right\} . (a\sqrt[3]{4} + k)$$
$$+ 4a^3 + 3a^2 (4)^{\frac{2}{3}} . k = 0,$$

whence, approximately,

$$3a^2 (2)^{\frac{2}{3}} . h - 3a \left\{a (4)^{\frac{1}{3}} h + a (2)^{\frac{1}{3}} k\right\} + 3a^2 (4)^{\frac{2}{3}} k = 0,$$

whence $$k = 0.$$

Thus $$\frac{dy}{dx} = -\frac{(a\sqrt[3]{2} + h)^2 - a^2 (4)^{\frac{1}{3}}}{a^2 (4)^{\frac{2}{3}} - a^2 (2)^{\frac{1}{3}}}$$

$$= -\frac{3a^2 (4)^{\frac{1}{3}} . h}{+}$$

$$= +, \text{ if } h \text{ be negative,}$$

$$= -, \text{ if } h \text{ be positive:}$$

hence $x = a\sqrt[3]{2}$ gives $a\sqrt[3]{4}$ as a maximum value of y.

If we take $y^3 = ax$, we shall get $x = 0$, $y = 0$, which we have already considered, and $x = a\sqrt[3]{4}$, $y = a\sqrt[3]{2}$, which makes $\frac{dy}{dx} = \infty$: we may shew, in precisely the same kind of way, that these values of the variables do not correspond to a maximum or minimum value of y.

Maxima and Minima of a Function of a Function.

76. Suppose that $r = f(x)$, and $x = \psi(\theta)$, $f(x)$ denoting a certain function of x, and $\psi(\theta)$ a function of θ; and let it be proposed to determine the maxima and minima of r as depending upon the variation of θ. We know that

$$\frac{dr}{d\theta} = \frac{dr}{dx} \cdot \frac{dx}{d\theta} = f'(x) \cdot \psi'(\theta).$$

From this relation it appears that, in order that r may have a maximum or minimum value, the expression

$$f'(x) \cdot \psi'(\theta)$$

must experience a change of sign as θ varies from $\theta_0 - a$ to $\theta_0 + a$, θ_0 being the value of θ which makes r a maximum or minimum, and a being an indefinitely small positive quantity. In order that such a change of sign may take place it is necessary that either $f'(x)$ or $\psi'(\theta)$ change sign, but that both do not change sign at once. In other words, that y may have a maximum or minimum value, it is necessary that either $f(x)$ have a maximum or minimum value as dependent upon the variation of x, or that $\psi(\theta)$ have a maximum or minimum value as dependent upon the variation of θ, and that $f(x)$ and $\psi(\theta)$ have not maximum or minimum values simultaneously. If $f(x)$ have a maximum or minimum value in relation to x, then r will have respectively a maximum or minimum value also in relation to θ: for, if $\psi'(\theta)$ be positive, then as θ varies from $\theta_0 - a$ to $\theta_0 + a$, x will vary from $x_0 - h$ to $x_0 + h$, and therefore, $f'(x)$ changing sign from + to −, $f'(x) \cdot \psi'(\theta)$ will also change sign from + to −; and, if $\psi'(\theta)$ be negative, then as θ varies from $\theta_0 - a$ to $\theta_0 + a$, x will vary from $x_0 + h$ to $x_0 - h$, and therefore, $f'(x)$ changing sign from

$-$ to $+$, $f'(x) \,.\, \psi'(\theta)$ will change sign from $+$ to $-$: that is, a maximum value of $f(x)$ with regard to x corresponds to a maximum value of r with regard to θ: we may shew in like manner that a minimum value of $f(x)$ with regard to x corresponds to a minimum value of r with regard to θ. Also, if $\psi(\theta)$ have a maximum value in relation to θ, r will have a maximum or a minimum value in relation to θ accordingly as $f'(x)$ is positive or negative; and if $\psi(\theta)$ have a minimum value in relation to θ, r will have a maximum or a minimum value in relation to θ accordingly as $f'(x)$ is negative or positive.

Ex. 1. Let it be proposed to find the maximum or minimum value of r, when

$$r = m + x, \quad x = m\left(\tan\frac{\theta}{2}\right)^2.$$

Here
$$\frac{dr}{dx} = 1,$$

a quantity insusceptible of a change of sign: hence r has no maximum or minimum in regard to x. Again

$$\frac{dx}{d\theta} = m\tan\frac{\theta}{2}\left(\sec\frac{\theta}{2}\right)^2 = m\,\frac{\sin\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}\right)^3}:$$

thus $\frac{dx}{d\theta}$ changes sign from $-$ to $+$ as θ varies from -0 to $+0$; and from $+$ to $-$ as θ varies from $\pi - \alpha$ to $\pi + \alpha$: hence

if $\theta = 0$, $x = 0$; $r = m$, a minimum value of r;

and if $\theta = \pi$, $x = +\infty$; $r = +\infty$, a maximum value of r.

This is the solution of the problem "to find the maximum or minimum values of the radius vector of a parabola, the focus being the pole."

Ex. 2. To find the maximum or minimum values of

$$r = \frac{b^2 x}{2a - x},$$

having given that
$$x = \frac{a(1 - e^2)}{1 + e\cos\theta}.$$

We have
$$\frac{dr}{dx} = \frac{2ab^2}{(2a - x)^2},$$

an essentially positive quantity, which shews that r has no maximum or minimum in relation to x taken absolutely.

Again $$\frac{dx}{d\theta} = \frac{ae\,(1 - e^2)}{(1 + e\cos\theta)^2} \,.\, \sin\theta,$$

which shews that $\frac{dx}{d\theta}$ changes sign from − to + as θ increases through zero, and from + to − as θ increases through π.

Hence

if $\theta = 0$, $x = a\,(1 - e)$; $r = \frac{1 - e}{1 + e} \,.\, b^2$, a minimum value of r;

and, if $\theta = \pi$, $x = a\,(1 + e)$; $r = \frac{1 + e}{1 - e} \,.\, b^2$, a maximum value of r.

This is the solution of the problem "to find the maximum value of the perpendicular drawn from the focus of an ellipse upon the tangent," r being the square of this distance and x the radius vector.

For additional examples the reader is referred to a paper in the *Cambridge Mathematical Journal*, for February, 1843, entitled "On certain cases of Geometrical Maxima and Minima."

Maxima and Minima of a Function of two Independent Variables.

77. Let $z = f(x, y)$, x and y being two independent variables. We are at liberty to assume that $y = \psi(x)$ provided that $\psi(x)$ denotes an arbitrary function of x. Differentiating z on this hypothesis, we have $$\frac{Dz}{dx} = \frac{dz}{dx} + \frac{dz}{dy} \,.\, y'$$

where y' is used to represent $\frac{dy}{dx}$.

Now that z may have a maximum or minimum value for any system of values x_0 and y_0 of x and y, it is sufficient and necessary that, as x increases through x_0, the total differential coefficient $\frac{Dz}{dx}$, and therefore the expression

$$\frac{dz}{dx} + \frac{dz}{dy} \cdot y',$$

shall change sign, whatever be the form of $\psi(x)$, the only restriction to the arbitrariness of this function being that it shall be equal to y_0 when $x = x_0$, and whatever therefore be the value of y'. A change of sign from plus to minus corresponds to a maximum, and, from minus to plus, to a minimum.

78. In what follows we shall confine our attention to those maxima and minima corresponding to which the partial differential coefficients of z do not assume either infinite or indeterminate values. Under this supposition we must have, for a maximum or minimum value of z,

$$\frac{Dz}{dx} = \frac{dz}{dx} + \frac{dz}{dy} \cdot y' = 0,$$

and therefore, y' being a perfectly arbitrary quantity,

$$\left.\begin{aligned} \frac{dz}{dx} &= 0 \\ \frac{dz}{dy} &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (1).$$

A pair of values of x and y, deducible from the equations (1), will certainly correspond to a maximum value of z, if

$$\frac{D^2z}{dx^2} < 0 \dots\dots\dots\dots\dots\dots (2),$$

and, to a minimum, if

$$\frac{D^2z}{dx^2} > 0 \dots\dots\dots\dots\dots\dots (3).$$

But, proceeding to the second total differential coefficient of z, we have, putting y'' for $\frac{d^2y}{dx^2}$,

$$\frac{D^2z}{dx^2} = \frac{d^2z}{dx^2} + 2\frac{d^2z}{dxdy} \cdot y' + \frac{d^2z}{dy^2} \cdot y'^2 + \frac{dz}{dy} \cdot y'',$$

or, by virtue of (1), when the values of x and y correspond to a maximum or minimum value of z,

$$\frac{D^2z}{dx^2} = \frac{d^2z}{dx^2} + 2\frac{d^2z}{dxdy} \cdot y' + \frac{d^2z}{dy^2} \cdot y'^2 \dots\dots\dots (4).$$

In order that, in conformity with the inequalities (2) and (3),

$\frac{D^2z}{dx^2}$ may never be zero whatever value be assigned to y', it is sufficient and necessary that the values of y' deducible from the quadratic equation

$$\frac{d^2z}{dx^2} + 2\frac{d^2z}{dxdy} \cdot y' + \frac{d^2z}{dy^2} \cdot y'^2 = 0$$

be impossible. Hence we must have

$$\frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} > \left(\frac{d^2z}{dxdy}\right)^2 \dots\dots\dots\dots\dots(5),$$

a condition which requires that $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{dy^2}$ have the same sign.

From (4) we see that

$$\frac{d^2z}{dy^2} \cdot \frac{D^2z}{dx^2} = \frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} - \left(\frac{d^2z}{dxdy}\right)^2 + \left(\frac{d^2z}{dy^2} \cdot y' + \frac{d^2z}{dxdy}\right)^2,$$

the right-hand member of which equation is, by virtue of the inequality (5), an essentially positive quantity. Hence $\frac{D^2z}{dx^2}$ must have the same sign as $\frac{d^2z}{dy^2}$ and therefore as $\frac{d^2z}{dx^2}$. Hence, to recapitulate, we see that, for a maximum or minimum value of z, it is necessary that

$$\frac{dz}{dx} = 0, \quad \frac{dz}{dy} = 0,$$

and that any pair of values of x and y, which satisfy these equations, and also the inequality

$$\frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} > \left(\frac{d^2z}{dxdy}\right)^2,$$

will certainly correspond to a maximum value of z, if $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{dy^2}$ be negative, and to a minimum, if $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{dy^2}$ be positive.

In the above reasoning we have supposed that $\frac{D^2z}{dx^2}$ is not equal to zero for values of x and y which satisfy the equations

(1): if however $\frac{D^2z}{dx^2} = 0$, then it will be sufficient that also $\frac{D^3z}{dx^3} = 0$, and that $\frac{D^4z}{dx^4} > 0$ or < 0; and so on, it being necessary that the first total differential coefficient of z which does not vanish be of an even order.

Suppose for instance that $\frac{D^4z}{dx^4}$ is the first of these coefficients which does not vanish. Then, for all values of y', we must have

$$\frac{D^2z}{dx^2} = \frac{d^2z}{dx^2} + 2\,\frac{d^2z}{dxdy}\,.\,y' + \frac{d^2z}{dy^2}\,.\,y'^2 + \frac{dz}{dy}\,.\,y'' = 0,$$

$$\frac{D^3z}{dx^3} = \frac{d^3z}{dx^3} + 3\,\frac{d^3z}{dx^2dy}\,.\,y' + 3\,\frac{d^3z}{dxdy^2}\,.\,y'^2 + \frac{d^3z}{dy^3}\,.\,y'^3$$
$$+ 3\left(\frac{d^2z}{dxdy} + \frac{d^2z}{dy^2}\,y'\right)y'' + \frac{dz}{dy}\,.\,y''' = 0,$$

where $y''' = \frac{d^3y}{dx^3}$; and therefore, in addition to the relation (1), we must have, y', y'', y''', being perfectly arbitrary quantities,

$$\frac{d^2z}{dx^2} = 0, \quad \frac{d^2z}{dxdy} = 0, \quad \frac{d^2z}{dy^2} = 0,$$

$$\frac{d^3z}{dx^3} = 0, \quad \frac{d^3z}{dx^2dy} = 0, \quad \frac{d^3z}{dxdy^2} = 0, \quad \frac{d^3z}{dy^3} = 0.$$

Since $\frac{D^4z}{dx^4}$ is supposed not to be zero, whatever be the value of y', it follows that the values of y' deducible from the biquadratic which results from the equation

$$\frac{D^4z}{dx^4} = 0,$$

viz.

$$\frac{d^4z}{dx^4} + 4\,\frac{d^4z}{dx^3dy}\,.\,y' + 6\,\frac{d^4z}{dx^2dy^2}\,.\,y'^2 + 4\,\frac{d^4z}{dxdy^3}\,.\,y'^3 + \frac{d^4z}{dy^4}\,.\,y'^4 = 0,$$

must be all impossible.

We may proceed in the same way to the consideration of cases in which the first total differential coefficient of z which does not vanish is of a higher order than the fourth.

Ex. 1. To find the maxima or minima of

$$z = x^3 + y^3 - 3axy.$$

Put
$$\left.\begin{aligned} \frac{dz}{dx} &= 3x^2 - 3ay = 0 \\ \frac{dz}{dy} &= 3y^2 - 3ax = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (1).$$

The equations (1) are satisfied by either of the systems

$$\begin{pmatrix} x = 0 \\ y = 0 \end{pmatrix} \text{ or } \begin{pmatrix} x = a \\ y = a \end{pmatrix}.$$

Differentiating again, we get

$$\frac{d^2z}{dx^2} = 6x, \quad \frac{d^2z}{dxdy} = -3a, \quad \frac{d^2z}{dy^2} = 6y.$$

If $x = 0$ and $y = 0$, then the expression

$$\frac{d^2z}{dx^2} \cdot \frac{d^2z}{dy^2} - \left(\frac{d^2z}{dxdy}\right)^2$$

has not a positive value, or this system of values does not correspond to either a maximum or a minimum value of u.

If $x = a$ and $y = a$, then the expression is equal to a positive quantity, and $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dy^2}$, are both equal to $6a$: hence, if a be positive, $-a^3$ is a minimum value of u, and, if a be negative, $-a^3$ is a maximum value of u.

Ex. 2. To ascertain whether $x = 0$, $y = 0$, which make $\frac{dz}{dx} = 0$ and $\frac{dz}{dy} = 0$, when

$$z = x^4 + x^2y^2 + y^4,$$

render z a maximum or a minimum.

We have

$$\frac{dz}{dx} = 4x^3 + 2xy^2 = 0,$$

$$\frac{dz}{dy} = 2x^2y + 4y^3 = 0,$$

$$\frac{d^2z}{dx^2} = 12x^2 + 2y^2 = 0,$$

$$\frac{d^2z}{dxdy} = 4xy = 0,$$

$$\frac{d^2z}{dy^2} = 2x^2 + 12y^2 = 0,$$

$$\frac{d^3z}{dx^3} = 24x = 0,$$

$$\frac{d^3z}{dx^2dy} = 4y = 0,$$

$$\frac{d^3z}{dxdy^2} = 4x = 0,$$

$$\frac{d^3z}{dy^3} = 24y = 0,$$

$$\frac{d^4z}{dx^4} = 24, \quad \frac{d^4z}{dx^3dy} = 0, \quad \frac{d^4z}{dx^2dy^2} = 4, \quad \frac{d^4z}{dxdy^3} = 0, \quad \frac{d^4z}{dy^4} = 24.$$

Hence the equation

$$\frac{d^4z}{dx^4} + 4\frac{d^4z}{dx^3dy}\cdot y' + 6\frac{d^4z}{dx^2dy^2}\cdot y'^2 + 4\frac{d^4z}{dxdy^3}\cdot y'^3 + \frac{d^4z}{dy^4}\cdot y'^4 = 0$$

becomes $$24 + 24y'^2 + 24y'^4 = 0,$$

the roots of which are evidently impossible.

Also $\frac{D^4z}{dx^4}$ is positive: hence it follows that $x = 0$, $y = 0$, correspond to a minimum value of z.

Maxima and Minima of Functions of any number of Independent Variables.

79. Suppose that $u = f(x, y, z, \dots)$

where $x, y, z, \dots$ are independent variables. Assume each of the variables $y, z, \dots$ to be an arbitrary function of x. Then, by the general theory of maxima and minima of a function of a single variable, it is necessary that the first of the total differential coefficients

$$\frac{Du}{dx}, \frac{D^2u}{dx^2}, \frac{D^3u}{dx^3}, \dots$$

which does not vanish shall be of an even order, positive for a minimum value of u, and negative for a maximum. If the n^{th} of these differential coefficients be the first which does not vanish, it is evident that, by adopting the same course of reasoning as in the case of two variables, we should ascertain that all the partial differential coefficients of u below the n^{th} must be equal to zero. In addition to this, we should find it to be necessary that a certain equation of n dimensions in $y', z', \ldots.$ the differential coefficients of $y, z, \ldots.$ with regard to x of which they are arbitrary functions, be incapable of being satisfied without assigning impossible values to at least some of the quantities $y', z' \ldots.$: the conditions arising from this consideration are generally of the utmost complexity.

An instance of Maxima and Minima corresponding to Indeterminate Differential Coefficients.

80. Let it be proposed to find the maximum or minimum value of

$$z = (x^2 + y^2)^{\frac{1}{3}}.$$

We have

$$\frac{dz}{dx} = \frac{2x}{3(x^2+y^2)^{\frac{2}{3}}}, \quad \frac{dz}{dy} = \frac{2y}{3(x^2+y^2)^{\frac{2}{3}}}.$$

When $x = 0$ and $y = 0$, the partial differential coefficients $\frac{dz}{dx}, \frac{dz}{dy}$, both assume the form $\frac{0}{0}$: we are unable therefore, by the tests of Art. (78), to ascertain whether this system of values corresponds to a maximum or minimum value of z. We shall have to consider whether $\frac{Dz}{dx}$, the total differential coefficient of z, has a change of sign, as x increases through zero, for all values of y'. Now

$$\frac{Dz}{dx} = \frac{dz}{dx} + \frac{dz}{dy} \cdot y' = \tfrac{2}{3} \cdot \frac{x + yy'}{(x^2+y^2)^{\frac{2}{3}}}.$$

Suppose that x increases from -0 to 0: then, if at the same time y increases from -0 to 0, y' must, by Art. (69), have always the sign $+$; while, if y decreases from $+0$ to 0, y' must have always the sign $-$: that is, if $x = -0$, then $x + yy' =$ a negative quantity. Next, let us suppose that x increases from 0 to $+0$: then, if, at the same time, y increases from 0 to $+0$, y' must have always the sign $+$: while, if y decreases from 0 to -0, y' must have always the sign $-$: that is, if $x = +0$, then $x + yy' =$ a positive quantity. We have shewn therefore that, as x increases through 0, $\frac{Dz}{dx}$ always changes sign from $-$ to $+$, whatever be the value of y'. Hence $x = 0$, $y = 0$, correspond to a minimum value of z, namely zero.

We might have treated this example more easily in the following manner.

Since $(x^2 + y^2)^{\frac{2}{3}}$ is essentially positive, we may instead of $\frac{Dz}{dx}$ take

$$\frac{Dv}{dx} = x + yy',$$

$\frac{Dv}{dx}$ being a function of x, y, y', which has always the same sign as $\frac{Dz}{dx}$. Putting

$$\frac{dv}{dx} = x = 0,$$

and

$$\frac{dv}{dy} = y = 0,$$

we see that

$$\frac{d^2v}{dx^2} = 1, \quad \frac{d^2v}{dx\,dy} = 0, \quad \frac{d^2v}{dy^2} = 1.$$

Hence

$$\frac{d^2v}{dx^2} \cdot \frac{d^2v}{dy^2} > \left(\frac{d^2v}{dx\,dy}\right)^2,$$

and $\frac{d^2v}{dx^2}$, $\frac{d^2v}{dy^2}$, have each a positive sign. Hence $x = 0$, $y = 0$, correspond to a minimum value of z.

The existence of maxima and minima of functions of two independent variables corresponding to values $\frac{0}{0}$ of the first partial differential coefficients of the function, was I believe first pointed out by M. J. Bertrand in Liouville's *Journal de Mathematiques,* tom. VIII. an. 1843, p. 155.

Application of Indeterminate Multipliers to Problems of Maxima and Minima.

81. Let $$u = f(x, y, z, \ldots\ldots),$$
a function of n variables $x, y, z, \ldots\ldots$ connected together by m equations

$$f_1(x, y, z, \ldots\ldots) = 0,\ f_2(x, y, z, \ldots\ldots) = 0,$$
$$f_3(x, y, z, \ldots\ldots) = 0, \ldots\ldots\ldots\ldots f_m(x, y, z, \ldots\ldots) = 0:$$

there will accordingly be $n - m$ independent variables. That u may be a maximum or minimum, we must have $Du = 0$, supposing that we confine ourselves to those maxima and minima which do not correspond to infinite or indeterminate values of the partial differential coefficients. Hence

$$\frac{df}{dx}\,dx + \frac{df}{dy}\,dy + \frac{df}{dz}\,dz + \ldots\ldots = 0 \ldots\ldots (1).$$

The equations connecting $x, y, z, \ldots\ldots$ give also

$$\left.\begin{array}{l}
\frac{df_1}{dx}\,dx + \frac{df_1}{dy}\,dy + \frac{df_1}{dz}\,dz + \ldots\ldots = 0 \\
\frac{df_2}{dx}\,dx + \frac{df_2}{dy}\,dy + \frac{df_2}{dz}\,dz + \ldots\ldots = 0 \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\frac{df_m}{dx}\,dx + \frac{df_m}{dy}\,dy + \frac{df_m}{dz}\,dz + \ldots\ldots = 0
\end{array}\right\} \ldots\ldots (2).$$

After eliminating m of the differentials dx, dy, $dz, \ldots\ldots$ between (1) and (2), $m + 1$ differential equations in all, we must equate separately to zero the multipliers of the $n - m$ remaining differentials which are entirely arbitrary; the $n - m$ equations thus obtained, together with the m equations connecting $x, y, z, \ldots\ldots$ will enable us to find the systems of values of $x, y, z, \ldots\ldots$ which will render u a maximum or minimum. It will then be necessary to ascertain whether $\frac{D^2u}{dx^2}$ becomes, by the substitution of these values, negative in the former and positive in the latter case, whatever relations be supposed to subsist among the $n - m$ arbitrary differentials. This examination of the sign of $\frac{D^2u}{dx^2}$ will usually involve very embarrassing computations: it frequently happens, however, especially in the applications of the theory of maxima and minima to questions of geometry and natural philosophy, that the existence of maxima and minima is certain from the nature of each particular problem, our only object being to ascertain the precise circumstances of such critical values. Under these circumstances, the examination of the sign of $\frac{D^2u}{dx^2}$, or, more generally, so as not to exclude cases where $\frac{Du}{dx} = \infty$ or $\frac{0}{0}$, the investigation whether $\frac{Du}{dx}$ necessarily changes sign as x passes through the critical value, becomes unnecessary.

The elimination of the arbitrary differentials may be effected very elegantly by the method of arbitrary multipliers. Multiply the equations (2) in order by the arbitrary quantities $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_m$, add the resulting equations to the equation (1); and then equate to zero the coefficients of all the differentials in the final equation. The legitimacy of this process will be evident when it is considered that by equating to zero the coefficients of the first m differentials we subject the m arbitrary factors $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_m$, to only m conditions, the coefficients of the remaining $n - m$ differentials being also necessarily zero by reason of the independency of these differentials. We thus get n equations

$$\frac{df}{dx} + \lambda_1 \frac{df_1}{dx} + \lambda_2 \frac{df_2}{dx} + \ldots\ldots + \lambda_m \frac{df_m}{dx} = 0,$$

$$\frac{df}{dy} + \lambda_1 \frac{df_1}{dy} + \lambda_2 \frac{df_2}{dy} + \ldots\ldots + \lambda_m \frac{df_m}{dy} = 0,$$

$$\frac{df}{dz} + \lambda_1 \frac{df_1}{dz} + \lambda_2 \frac{df_2}{dz} + \ldots\ldots + \lambda_m \frac{df_m}{dz} = 0,$$

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

We have therefore in all $n+m$ equations involving the n variables $x, y, z, \ldots\ldots$ and the m arbitrary multipliers $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_m$: whence $x, y, z, \ldots\ldots$ and therefore u may be determined.

Ex. 1. Suppose that

$$u = x^2 + y^2 + z^2 + \ldots\ldots,$$

and that the variables $x, y, z, \ldots$ are connected by the equation

$$ax + by + cz + \ldots\ldots = k,$$

$a, b, c, \ldots\ldots k$, being constant quantities. Then, by the theory laid down, we have

$$x + a\lambda_1 = 0, \quad y + b\lambda_1 = 0, \quad z + c\lambda_1 = 0, \ldots\ldots,$$

which equations are equivalent to

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence, by the ordinary theories of proportion,

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \ldots\ldots = \frac{x^2 + y^2 + z^2 + \ldots\ldots}{a^2 + b^2 + c^2 + \ldots\ldots} = \frac{u}{a^2 + b^2 + c^2 + \ldots\ldots};$$

and also $\dfrac{x^2}{ax} = \dfrac{y^2}{by} = \dfrac{z^2}{cz} = \ldots\ldots = \dfrac{x^2 + y^2 + z^2 + \ldots\ldots}{ax + by + cz + \ldots\ldots} = \dfrac{u}{k}$:

hence $$\frac{u}{a^2 + b^2 + c^2 + \ldots\ldots} = \frac{u^2}{k^2},$$

or $$u = \frac{k^2}{a^2 + b^2 + c^2 + \ldots\ldots}.$$

We may easily satisfy ourselves that this is really a minimum value of u. In fact there is, identically,

$$(a^2 + b^2 + c^2 + \ldots\ldots) \cdot (x^2 + y^2 + z^2 + \ldots\ldots)$$

$$= (ax + by + cz + \ldots\ldots)^2 + (bx - ay)^2 + (cx - az)^2 + \ldots\ldots,$$

and therefore

$$u = \frac{k^2 + (bx - ay)^2 + (cx - az)^2 + \ldots\ldots}{a^2 + b^2 + c^2 + \ldots\ldots},$$

which shews that the inequality

$$u > \frac{k^2}{a^2 + b^2 + c^2 + \ldots\ldots}$$

is verified for all values of $x, y, z, \ldots.$ which do not satisfy the relations (1).

Ex. 2. To find the maximum or minimum of the function

$$u = x^p y^q z^r \ldots. ,$$

where $p, q, r, \ldots.$ are all positive quantities, the variables $x, y, z, \ldots.$ being subject to the equation

$$ax + by + cz + \ldots. = k.$$

In this case we have, putting $D \log u = \frac{Du}{u} = 0$,

$$\frac{p\,dx}{x} + \frac{q\,dy}{y} + \frac{r\,dz}{z} + \ldots. = 0,$$

and $$a\,dx + b\,dy + c\,dz + \ldots. = 0,$$

and therefore

$$\frac{p}{x} + \lambda a = 0, \quad \frac{q}{y} + \lambda b = 0, \quad \frac{r}{z} + \lambda c = 0, \ldots.,$$

whence $$\frac{p}{ax} = \frac{q}{by} = \frac{r}{cz} = \ldots. = \frac{p+q+r+\ldots.}{k},$$

$$x = \frac{p}{a} \cdot \frac{k}{p+q+r+\ldots.},$$

$$y = \frac{q}{b} \cdot \frac{k}{p+q+r+\ldots.},$$

$$z = \frac{r}{c} \cdot \frac{k}{p+q+r+\ldots.},$$

. . . .

. . . .

We have also

$$\frac{Du}{u} = \frac{p\,dx}{x} + \frac{q\,dy}{y} + \frac{r\,dz}{z} + \ldots.,$$

and therefore, considering $dx, dy, dz, \ldots\ldots$ constant, while $x, y, z, \ldots.$ are supposed to vary, as we are at liberty to do in this example because there is no equation connecting the general values of $x, y, z, \ldots.$ and their differentials,

$$\frac{D^2u}{u} - \left(\frac{Du}{u}\right)^2 = -p\left(\frac{dx}{x}\right)^2 - q\left(\frac{dy}{y}\right)^2 - r\left(\frac{dz}{z}\right)^2 - \ldots..$$

or, since $Du = 0$,

$$\frac{D^2u}{u} = -\left\{p\left(\frac{dx}{x}\right)^2 + q\left(\frac{dy}{y}\right)^2 + r\left(\frac{dz}{z}\right)^2 + \dots\right\},$$

which shews that D^2u is essentially negative. The values of $x, y, z, \dots$ therefore, which we have found above, correspond to a maximum value of u.

Ex. 3. To find the minimum value of

$$u = x^2 + y^2 + z^2 \;\dots\dots\dots\dots\; (1),$$

x, y, z, being subject to the conditions

$$ax + by + cz = 1 \;\dots\dots\dots\dots\; (2),$$

$$a'x + b'y + c'z = 1 \;\dots\dots\dots\dots\; (3).$$

It is evident that there is a minimum value of u, for if we were to suppose $u = 0$, then we should have $x = 0$, $y = 0$, $z = 0$, which values of the variables are incompatible with the equations (2) and (3).

Differentiating (1), (2), (3), and putting $Du = 0$, we get

$$x\,dx + y\,dy + z\,dz = 0,$$

$$a\,dx + b\,dy + c\,dz = 0,$$

$$a'\,dx + b'\,dy + c'\,dz = 0:$$

multiplying these three equations in order by $1, \lambda, \lambda'$, adding, and equating to zero the coefficients of the differentials, we have

$$x + \lambda a + \lambda' a' = 0 \;\dots\dots\dots\dots\; (4),$$

$$y + \lambda b + \lambda' b' = 0 \;\dots\dots\dots\dots\; (5),$$

$$z + \lambda c + \lambda' c' = 0 \;\dots\dots\dots\dots\; (6).$$

Multiplying (4), (5), (6), by x, y, z, respectively, and adding, we get by virtue of (1), (2), (3),

$$u + \lambda + \lambda' = 0 \;\dots\dots\dots\dots\; (7):$$

multiplying them by a, b, c, and adding, we get

$$1 + \lambda(a^2 + b^2 + c^2) + \lambda'(aa' + bb' + cc') = 0 \;\dots\; (8):$$

multiplying them by a', b', c', and adding, we get

$$1 + \lambda(aa' + bb' + cc') + \lambda'(a'^2 + b'^2 + c'^2) = 0 \;\dots\; (9).$$

From (8) and (9) we have

$$0 = aa' + bb' + cc' - (a^2 + b^2 + c^2)$$
$$+ \lambda' \{(aa' + bb' + cc')^2 - (a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2)\}$$

and $0 = aa' + bb' + cc' - (a'^2 + b'^2 + c'^2)$

$$+ \lambda \{(aa' + bb' + cc')^2 - (a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2)\},$$

whence $0 = -(a - a')^2 - (b - b')^2 - (c - c')^2$

$$+ (\lambda + \lambda')\{-(ab' - a'b)^2 - (bc' - b'c)^2 - (ca' - c'a)^2\},$$

and therefore, from (7),

$$u = \frac{(a - a')^2 + (b - b')^2 + (c - c')^2}{(ab' - a'b)^2 + (bc' - b'c)^2 + (ca' - c'a)^2}.$$

CHAPTER VII.

DEVELOPMENT OF FUNCTIONS.

Taylor's Theorem.

82. Let $y = f(x)$, and when for x we substitute successively $x + \delta x$, $x + 2\delta x$, $x + 3\delta x$, ... let the corresponding values of y be denoted by y_1, y_2, y_3, ... Then, by Art. (46), we have

$$y_n = f(x + n\delta x) = y + \frac{n}{1}\delta y + \frac{n(n-1)}{1.2}\delta^2 y + \dots + \frac{n}{1}\delta^{n-1}y + \delta^n y,$$

and therefore, putting $n\delta x = h$,

$$f(x+h) = f(x) + \frac{h}{1}\cdot\frac{\delta f(x)}{\delta x} + \frac{h^2}{1.2}\left(1 - \frac{1}{n}\right)\frac{\delta^2 f(x)}{\delta x^2}$$

$$+ \frac{h^3}{1.2.3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{\delta^3 f(x)}{\delta x^3} + \dots$$

$$+ \frac{h^n}{1.2.3\dots n}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right)\dots\left(1 - \frac{n-1}{n}\right)\frac{\delta^n f(x)}{\delta x^n}\dots(1).$$

Suppose now the increment δx to be indefinitely diminished, the number n being at the same time indefinitely increased, so that the product $n\delta x = h$ may remain equal to a finite quantity. The number of the terms in the right-hand member of the equation will be indefinitely increased, and the ratios

$$\frac{\delta f(x)}{\delta x}, \quad \frac{\delta^2 f(x)}{\delta x^2}, \quad \frac{\delta^3 f(x)}{\delta x^3}, \dots.$$

will converge to the limits

$$\frac{df(x)}{dx},\ \frac{d^2f(x)}{dx^2},\ \frac{d^3f(x)}{dx^3},\ldots.$$

or

$$f'(x),\ f''(x),\ f'''(x),\ldots.$$

moreover the fractions

$$1-\frac{1}{n},$$

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right),$$

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right),$$

. . . .

. . . .

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\ldots\left(1-\frac{r-1}{n}\right),$$

(where r is any positive integer less than n, which remains constant while n increases,) will all converge to unity. Hence, whatever be the magnitude of r, and whatever value be assigned to h, n may be taken so large that the sum of the first $r + 1$ terms of the second member of the equation (1) will approach without limit to the expression

$$f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1.2}f''(x)+\frac{h^3}{1.2.3}f'''(x)+\ldots.+\frac{h^r}{1.2.3\ldots r}f^r(x)\ldots(2).$$

We suppose that all the functions $f(x), f'(x), f''(x), f'''(x), \ldots.\ f^r(x)$, have finite values, an hypothesis without which the preceding reasoning would be inconclusive: in fact the product of the expression

$$\frac{h^r}{1.2.3.\ldots.r}\,\frac{\delta^r f(x)}{\delta x^r}$$

in (1) and the defect of its coefficient from unity, would assume the form $0 \times \infty$, a quantity not necessarily zero.

We are not at liberty, from the conclusions already established, to regard the function $f(x + h)$ as equivalent to the sum of the series (2) continued to an infinite number of terms. For the $(\nu+1)^{\text{th}}$ term in the series (1), supposing ν to be a number, between r and n, much greater than r, and, in fact, comparable with n, will not converge to the expression

$$\frac{h^r}{1.2.3.\ldots.\nu} f_\nu(x),$$

because the factor

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots.\left(1-\frac{\nu-1}{n}\right)$$

will not converge to unity, but, in fact, for values of ν very nearly in a ratio of equality to n, will be reduced to zero.

We may however always suppose that

$$f(x+h)=f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1.2}f''(x)+\frac{h^3}{1.2.3}f'''(x)+\ldots.$$
$$+\frac{h^r}{1.2.3\ldots r}f^r(x)+R_r,$$

R_r being an unknown function of x and h. If we determine the conditions that R_r may lie between limits, which, as r increases indefinitely, become less than any assignable quantities, we shall have thereby ascertained the conditions under which the function $f(x+h)$ may be considered to be equivalent to the series (2) continued to infinity.

83. With this object in view, we remark that, if a function $\phi(h)$ is equal to zero when $h=0$, and if its derivative $\phi'(h)$ retains always the same sign between the limits 0, h, without passing through infinity, the function $\phi(h)$ will have always for this interval the same sign as $\phi'(h)$. For

$$\phi'(h) = \text{the limit of } \frac{\delta\phi(h)}{\delta h},$$

and therefore $\delta\phi(h)$, when δh is indefinitely small, must have always the same sign as $\phi'(h)\,\delta h$, that is, $\phi(h)$ must, with the continuous increase of h, be continuously increasing or continuously diminishing accordingly as $\phi'(h)$ is positive or negative, but, by hypothesis, $\phi(h)=0$ when $h=0$; it follows therefore that $\phi(h)$ must always have the same sign as $\phi'(h)$.

Suppose now that we assume, as we are evidently at liberty to do, that

$$R_r = \frac{h^{r+1}}{1.2.3 \ldots . (r+1)} \cdot \psi(x, h),$$

ψ being a symbol of unknown functionality. It is manifest then that we shall have determined inferior and superior limits to the value of R_r, if we ascertain two quantities P, Q, such that, for all values of h,

$$\left.\begin{array}{l} f(x+h) - \left\{ f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1.2} f''(x) + \frac{h^3}{1.2.3} f'''(x) \right. \\ \quad \left. + \ldots . + \frac{h^r}{1.2.3 \ldots r} f^r(x) + \frac{h^{r+1}}{1.2.3 \ldots (r+1)} \cdot P \right\} > 0, \\ f(x+h) - \left\{ f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1.2} f''(x) + \frac{h^3}{1.2.3} f'''(x) \right. \\ \quad \left. + \ldots . + \frac{h^r}{1.2.3 \ldots r} f^r(x) + \frac{h^{r+1}}{1.2.3 \ldots (r+1)} \cdot Q \right\} < 0, \end{array} \right\} \ldots (3).$$

Let $\phi_1(h)$, $\phi_2(h)$, represent the former members of these two inequalities: the functions $\phi_1(h)$, $\phi_2(h)$, both vanish when $h = 0$; hence, from what has been said above, we know that the inequalities will be satisfied if we have, for all values of h,

$$\phi_1'(h) > 0, \quad \phi_2'(h) < 0,$$

or

$$\left.\begin{array}{l} f'(x+h) - \left\{ f'(x) + \frac{h}{1} f''(x) + \frac{h^2}{1.2} f'''(x) + \ldots . \right. \\ \quad \left. + \frac{h^{r-1}}{1.2.3 \ldots (r-1)} f^r(x) + \frac{h^r}{1.2.3 \ldots r} \cdot P \right\} > 0, \\ f'(x+h) - \left\{ f'(x) + \frac{h}{1} f''(x) + \frac{h^2}{1.2} f'''(x) + \ldots . \right. \\ \quad \left. + \frac{h^{r-1}}{1.2.3 \ldots (r-1)} \cdot f^r(x) + \frac{h^r}{1.2.3 \ldots r} \cdot Q \right\} < 0 \end{array} \right\} \ldots (4).$$

But $\phi_1'(h)$, $\phi_2'(h)$, are also functions of h which vanish when $h = 0$: hence the inequalities (4), and consequently the inequalities (3), will be satisfied if

$$\phi_1''(h) > 0, \quad \phi_2''(h) < 0,$$

$$\left.\begin{aligned}&\text{or } f''(x+h)-\Big\{f''(x)+\frac{h}{1}f'''(x)+\dots\dots\\&\qquad+\frac{h^{r-2}}{1.2.3\dots(r-2)}\cdot f^{r}(x)+\frac{h^{r-1}}{1.2.3\dots(r-1)}\cdot P\Big\}>0,\\&f''(x+h)-\Big\{f''(x)+\frac{h}{1}f'''(x)+\dots\dots\\&\qquad+\frac{h^{r-2}}{1.2.3\dots(r-2)}\cdot f^{r}(x)+\frac{h^{r-1}}{1.2.3\dots(r-1)}\cdot Q\Big\}<0.\end{aligned}\right\}\dots(5).$$

Continuing this reasoning, it is evident that after $r+1$ successive differentiations, we shall arrive at the inequalities

$$\left.\begin{aligned}f^{r+1}(x+h)-P>0\\f^{r+1}(x+h)-Q<0\end{aligned}\right\}\dots\dots\dots\dots(6).$$

Now these final inequalities, and consequently the primitive inequalities (3), will be satisfied, if we assign to P and Q respectively the least and the greatest value assumed by the derived function

$$f^{r+1}(z),$$

for the values of z which lie between x and $x+h$. Hence

$$R_r=\frac{h^{r+1}}{1.2.3\dots(r+1)}\cdot f^{r+1}(x+\theta h),$$

θ denoting an unknown numerical quantity lying between 0 and 1.

Hence we finally conclude that

$$\begin{aligned}f(x+h)=f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1.2}f''(x)+\frac{h^3}{1.2.3}f'''(x)+\dots\\+\frac{h^r}{1.2.3\dots r}f^{r}(x)+\frac{h^{r+1}}{1.2.3\dots(r+1)}\cdot f^{r+1}(x+\theta h)\dots(7).\end{aligned}$$

Now, if the numerical values of the function $f^{r+1}(z)$ never exceed a certain finite magnitude λ, for all values of z from $z=x$ to $z=x+h$, and for all values of r the index of differentiation, it is plain that

$$R_r<\frac{h^{r+1}\,.\,\lambda}{1.2.3\dots(r+1)};$$

and then, whatever be the value of h, we may always take r so large that R_r may become less than any assignable magnitude. Hence, in this case, the function $f(x+h)$ will be equal to the series (2) continued to infinity, or

$$f(x+h) = f(x) + \frac{h}{1} f'(x) + \frac{h^2}{1.2} f''(x) + \frac{h^3}{1.2.3} f'''(x) + \ldots$$

This formula is commonly called Taylor's theorem, after the name of its discoverer, Brook Taylor; it was published in the year 1715, in his *Methodus Incrementorum.*

Another Demonstration of Taylor's Theorem.

84. By the application of the elementary principles of integration, a demonstration of Taylor's formula may be given, which has the advantage of determining not only the limits of R_r, but also its actual value expressed by a definite integral. The student may pass over this demonstration until he has become acquainted with the Integral Calculus.

Whatever be the function f, provided that it remains continuous between the values of the variable designated by x and $x + h$, we shall have

$$f(x+h) - f(x) = \int_x^{x+h} f'(x) \,.\, dx.$$

In order to express more clearly the distinction between the general value of x under the sign of integration, and the value of x which enters into the expression of the limits, we may write

$$f(x+h) - f(x) = \int_x^{x+h} f'(x') \,.\, dx',$$

or, putting $x' = x + h - z$, and therefore $dx' = -dz$,

$$\begin{aligned} f(x+h) - f(x) &= -\int_h^0 f'(x+h-z)\, dz \\ &= \int_0^h f'(x+h-z)\, dz. \end{aligned}$$

But, integrating by parts,

$$\int_0^h f'(x+h-z)\, dz = hf'(x) + \int_0^h z f''(x+h-z)\, dz;$$

hence $f(x+h) = f(x) + \frac{h}{1} f'(x) + \int_0^h z f''(x+h-z)\, dz.$

Again, integrating by parts,

$$\int_0^h z f''(x+h-z)\, dz = \tfrac{1}{2} h^2 f''(x) + \tfrac{1}{2} \int_0^h z^2 f'''(x+h-z)\, dz;$$

hence

$$f(x+h)=f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1.2}f''(x)+\frac{1}{1.2}\int_0^h z^2 f'''(x+h-z)\,dz.$$

Similarly we may shew that

$$f(x+h)=f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1.2}f''(x)+\frac{h^3}{1.2.3}f'''(x)$$
$$+\frac{1}{1.2.3}\int_0^h z^3 f^{\text{IV}}(x+h-z)\,dz,$$

and generally, by a series of successive steps, we see that

$$f(x+h)=f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1.2}f''(x)+\frac{h^3}{1.2.3}f'''(x)+\dots$$
$$+\frac{h^r}{1.2.3\dots r}f^r(x)+\frac{1}{1.2.3\dots r}\int_0^h z^r f^{r+1}(x+h-z)\,dz.$$

Now, supposing P and Q to represent the least and the greatest value of the function

$$f^{r+1}(x+h-z),$$

for the series of values of z comprised between 0 and h, it is evident that we shall have

$$\int_0^h z^r f^{r+1}(x+h-z)\,dz > \left\{P\int_0^h z^r\,dz=\frac{Ph^{r+1}}{r+1}\right\}$$
$$\text{and} < \left\{Q\int_0^h z^r\,dz=\frac{Qh^{r+1}}{r+1}\right\}.$$

It appears therefore that the value of R_r falls between

$$\frac{Ph^{r+1}}{1.2.3\dots(r+1)} \text{ and } \frac{Qh^{r+1}}{1.2.3\dots(r+1)}:$$

and that moreover

$$R_r=\frac{1}{1.2.3\dots r}\int_0^h z^r f^{r+1}(x+h-z)\,dz,$$

which gives the actual value of R_r.

Cauchy's Expression for R_r.

85. Putting z for x, and then replacing h by $x-z$ in the formula (7), we get

$$f(x) = f(z) + \frac{x-z}{1} f'(z) + \frac{(x-z)^2}{1.2} f''(z) + \dots$$
$$+ \frac{(x-z)^r}{1.2.3\dots r} f^r(z) + \phi(z) \;\dots\dots\; (8),$$

where $$\phi(z) = \frac{(x-z)^{r+1}}{1.2.3\dots(r+1)} f^{r+1}\{z + \theta(x-z)\}.$$

When $r = 0$, the formula (8) gives

$$f(x) = f(z) + (x-z) f'\{z + \theta_1(x-z)\},$$

θ_1 denoting a number between 0 and 1, not generally the same as θ: hence, replacing f by ϕ, and observing that, by (8), $\phi(x) = 0$, we have

$$0 = \phi(z) + (x-z)\phi'\{z + \theta_1(x-z)\} \;\dots\dots\; (9).$$

Again, by differentiating (8) with regard to z, we have, cancelling terms in the result which destroy each other,

$$0 = \frac{(x-z)^r}{1.2.3\dots r} . f^{r+1}(z) + \phi'(z),$$

whence, putting $z + \theta_1(x-z)$ for z,

$$\phi'\{z + \theta_1(x-z)\} = -\frac{(1-\theta_1)^r (x-z)^r}{1.2.3\dots r} f^{r+1}\{z + \theta_1(x-z)\}.$$

Consequently we have, from (9), since $\phi(x) = 0$,

$$\phi(z) = \frac{(1-\theta_1)^r (x-z)^{r+1}}{1.2.3\dots r} . f^{r+1}\{z + \theta_1(x-z)\},$$

and therefore we may replace, in the formula (7), the expression for the remainder R_r, viz.

$$\frac{h^{r+1}}{1.2.3\dots(r+1)} . f^{r+1}(x + \theta h),$$

by this equivalent expression

$$\frac{(1-\theta_1)^r . h^{r+1}}{1.2.3\dots r} . f^{r+1}(x + \theta_1 h).$$

Examples of Taylor's Theorem.

86. Ex. 1. Let $f(x) = \log x$.

Then

$$\frac{df(x)}{dx} = \frac{1}{x}, \quad \frac{d^2f(x)}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3f(x)}{dx^3} = \frac{2}{x^3}, \quad \frac{d^4f(x)}{dx^4} = -\frac{2.3}{x^4}, \dots.$$

and therefore

$$f(x+h) = \log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \dots.$$

Ex. 2. Let $f(x) = \sin x$: then

$$\frac{df(x)}{dx} = \cos x, \quad \frac{d^2f(x)}{dx^2} = -\sin x, \quad \frac{d^3f(x)}{dx^3} = -\cos x,$$

$$\frac{d^4f(x)}{dx^4} = \sin x, \dots,$$

and therefore

$$f(x+h) = \sin(x+h) = \sin x + \frac{h}{1} \cdot \cos x - \frac{h^2}{1.2} \sin x - \frac{h^3}{1.2.3} \cdot \cos x$$
$$+ \frac{h^4}{1.2.3.4} \sin x - \&c.$$

Failure of Taylor's Theorem.

87. In the demonstration of Taylor's theorem we have supposed that the function $f(x)$ and its derivatives are all of them finite. If, for a particular value a of x, this should not be the case, the theorem then becomes inapplicable, or is said to fail. Suppose for instance that $f(x)$ involves a term of either of the following forms

$$\phi(x) \,.\, (x-a)^{-m} \dots\dots\dots\dots\dots (1),$$

$$\phi(x) \,.\, (x-a)^{n+\omega} \dots\dots\dots\dots\dots (2),$$

m and n being positive integers, and ω a proper positive fraction.

In the case of the form (1), $f(x)$ itself becomes infinite when $x = a$, and, in the case of the form (2), all its derived functions after the n^{th}. These two forms comprehend the only cases of failure which can present themselves in ordinary algebraical functions.

Suppose that $x = a + h$; then

$$\phi(x).(x-a)^{-m} = \phi(a+h).h^{-m}$$
$$= \left\{\phi(a) + \frac{h}{1}\phi'(a) + \frac{h^2}{1.2}\phi''(a) + \ldots\right\} h^{-m},$$

which shews that a failure of the theorem, due to a term in $f(x)$ of the form (1), corresponds to the existence of negative powers of h in the true development of $f(a+h)$.

Putting $x = a + h$ in (2), we see that

$$\phi(x).(x-a)^{n+\omega} = \phi(a+h).h^{n+\omega}$$
$$= \left\{\phi(a) + \frac{h}{1}\phi'(a) + \frac{h^2}{1.2}\phi''(a) + \ldots\right\} h^{n+\omega},$$

which shews that a failure, arising from a term of the form (2), corresponds to the existence of fractional powers of h in the true development of $f(a+h)$.

88. When the first derived function of $f(x)$ which becomes infinite for the particular value a of x, is of the $(r+2)^{\text{th}}$ order, we may employ Taylor's development provided that we do not carry it beyond the term involving h^r, and that we take care to preserve the remainder R_r which may be evaluated by the formula of Art. (84), a formula which is always applicable when the $(r+1)^{\text{th}}$ derivative, as we are now supposing, remains finite between the limits.

Lagrange's Theory of Functions.

89. The extreme generality of Taylor's series had for a long time attracted the attention of analysts, when Lagrange conceived the idea of adopting it as the basis of a theory of functions, the object of which was to arrive at the conclusions of the differential calculus without introducing the idea of limits or infinitesimal quantities. We will give a brief sketch of the general system of Lagrange, which has been, until within the last few years, so generally adopted in elementary treatises.

Lagrange assumes, in the first place, that any algebraical function $f(x+h)$ can be developed in a series of the form

$$f(x) + \phi_1(x) \,.\, h^\alpha + \phi_2(x) \,.\, h^\beta + \phi_3(x) \,.\, h^\gamma + \ldots.$$

He then proceeds to shew that, from the algebraical nature of functions, these powers of h can neither be fractional nor negative, so long as x and h remain general in form. He remarks that, if the series were to involve a fractional power of h, this term would have several values, and that accordingly, for a single system of values of x and $f(x)$, the function $f(x+h)$ would have several distinct values; a conclusion which cannot be true for all values of x and $f(x)$, precisely as, to borrow an illustration from algebraic geometry, it is impossible that all the points of a curve should be multiple points. Again, if the series were to involve negative powers of h, he observes that the corresponding terms would become infinite when $h = 0$, and that consequently $f(x)$ would be always infinite, which is impossible, except for particular values of x.

From the above reasoning, then, he concludes that the development of $f(x+h)$ is expressed in a general form by the equation

$$f(x+h) = f(x) + h\,\phi_1(x) + h^2\,\phi_2(x) + h^3\,\phi_3(x) + \ldots.$$

The coefficient $\phi_1(x)$ of h, Lagrange calls the *derived function* or *derivative* of $f(x)$, and represents by the expression $f'(x)$. Having thus defined the method of derivation, he determines without difficulty the law by which the functions $\phi_2(x)$, $\phi_3(x)$,... are derived from $f'(x)$, and thus arrives at the formula of Taylor,

$$f(x+h) = f(x) + \frac{h}{1}\,f'(x) + \frac{h^2}{1.2}\,f''(x) + \frac{h^3}{1.2.3}\,f'''(x) + \ldots,$$

each of the successive functions $f'(x)$, $f''(x)$, $f'''(x)$,.... being derived from the preceding just as $f'(x)$ is derived from $f(x)$. These functions are termed the first, second, third, &c. derivatives of $f(x)$, the order of derivation being indicated by the dashes.

If then in the elementary functions x^m, log x, sin x, &c. we substitute $x+h$ for x, and then expand, by the ordinary processes of algebra, $(x+h)^m$, $\log(x+h)$, $\sin(x+h)$, &c., in series ascending by positive integral powers of h, the coefficients of the first power of h in these developments will be the first derivatives of x^m, $\log x$, $\sin x$, &c. The second derivatives may in the same way be derived from the first, and so on indefinitely to higher orders of derivation. The derivatives of composite functions, which depend upon those of the elementary functions, may then be determined.

The theory of functions therefore, according to the method of Lagrange, resolves itself into a mere algebraical system, to the exclusion of what he considers to be the extraneous idea of infinitesimals or, in the language of Newton, fluxions. For the complete development of this theory the reader is referred to Lagrange's two systematic Treatises, entitled *Théorie des fonctions analytiques* and *Leçons sur le calcul des fonctions.*

Within the last few years the logical value of Lagrange's system has been called into question by all writers of authority both in France and in England. The chief objections to his method may be arranged under four heads.

(1) All inductions drawn from developments in the form of divergent series are devoid of solidity, and frequently, as may be ascertained from actual instances, lead to erroneous results. It would appear therefore that Lagrange's method, as involving the consideration of series without regard to convergency, is not entitled to the reputation, which it originally possessed, of being rigorously logical.

(2) The hypothesis that $f(x+h)$ may be expanded in a series by positive integral powers of h, restricts the application of Lagrange's system to the ordinary functions of algebra, whereas the general theory of limits embraces all continuous functions whatever, and involves a code of doctrine which exists independently of its application to any subordinate science.

(3) In the application of the theory of functions to geometry and natural philosophy, the idea of limits cannot really be avoided, although it may be disguised by the artifices of algebra. The doctrine of limits on the other hand, as explicitly involving in an abstract form essential principles of our conceptions of

curvature and of motion, lies in immediate contact with its most interesting applications.

(4) When the remainder R_r keeps indefinitely diminishing without limit as r increases, Taylor's series is convergent; the converse proposition is not, however, generally true. The theorem therefore, regarded as an indefinite expansion, fails to express the true value of $f(x+h)$, not only whenever the series is divergent, but also, which constitutes a still greater limitation, whenever R_r does not diminish indefinitely with the increase of r.

Stirling's Theorem.

90. If in the formula of Taylor we put $x = 0$, we get

$$f(h) = f(0) + \frac{h}{1} \cdot f'(0) + \frac{h^2}{1.2} f''(0) + \frac{h^3}{1.2.3} f'''(0) + \dots,$$

or, putting x in place of h,

$$f(x) = f(0) + \frac{x}{1} \cdot f'(0) + \frac{x^2}{1.2} f''(0) + \frac{x^3}{1.2.3} f'''(0) + \dots.$$

This theorem enables us to expand any function of x in a series ascending by positive integral powers of x.

Its demonstration was given by Stirling in a work entitled *Methodus Differentialis*, London, 1730, p. 102. The theorem was afterwards given by Maclaurin in his *Treatise of Fluxions*, Edinburgh, 1742, p. 610; and is now generally called Maclaurin's theorem. It is in fact, as we see from the demonstration given above, merely a particular case of Taylor's theorem.

Stirling's series, as well as Taylor's, from which it has been deduced, ought to be completed by a remainder.

Let R_r' represent the remainder, when we stop at the $(r+1)^{\text{th}}$ term of the development: then

$$f(x) = f(0) + \frac{x}{1} f'(0) + \frac{x^2}{1.2} f''(0) + \frac{x^3}{1.2.3} f'''(0) + \dots\dots$$
$$+ \frac{x^r}{1.2.3.\dots.r} \cdot f^r(0) + R_r',$$

the value of R_r' being given in the form of a definite integral by the equation

$$R_r' = \frac{1}{1.2.3.\ldots.r}\int_0^x z^r f^{r+1}(x-z)\,dz,$$

a formula derived from the expression for R^r in Art. (84), by first putting $x = 0$, and then replacing h by x in the result.

Again, the limits between which R_r' always lies, are

$$\frac{P'x^{r+1}}{1.2.3.\ldots.(r+1)} \text{ and } \frac{Q'x^{r+1}}{1.2.3.\ldots.(r+1)},$$

P', Q', denoting respectively the least and the greatest values of $f^{r+1}(x-z)$, or, which amounts to the same thing, of $f^{r+1}(z)$, between the limits $z = 0$, $z = x$.

If θ, θ_1, represent certain unknown numbers comprised between 0 and 1, we shall have also for R_r' the two expressions

$$R_r' = \frac{x^{r+1}}{1.2.3.\ldots.(r+1)}\cdot f^{r+1}(\theta x),\quad R_r' = \frac{(1-\theta_1)^r x^{r+1}}{1.2.3.\ldots.r}\cdot f^{r+1}(\theta_1 x).$$

Examples of the Application of Stirling's Theorem.

91. Ex. 1. Suppose that $f(x) = a^x$.

Then

$$f'(x) = \log a\,.\,a^x,\quad f''(x) = (\log a)^2.\,a^x,\quad f'''(x) = (\log a)^3.\,a^x,\ldots.$$

hence, putting $x = 0$,

$$f'(0) = \log a,\quad f''(0) = (\log a)^2,\quad f'''(0) = (\log a)^3,\ldots\ldots$$

We have then, by Stirling's theorem,

$$a^x = 1 + \frac{x}{1}\log a + \frac{x^2}{1.2}(\log a)^2 + \frac{x^3}{1.2.3}(\log a)^3 + \ldots + \frac{x^r}{1.2.3\ldots r}(\log a)^r$$
$$+ \frac{x^{r+1}}{1.2.3.\ldots.(r+1)}\cdot(\log a)^{r+1}.\,a^{\theta x}.$$

The remainder

$$\frac{\{x\log(a)\}^{r+1}}{1.2.3.\ldots.(r+1)}\cdot a^{\theta x},$$

which is equal to

$$\frac{x \log a}{1} \cdot \frac{x \log a}{2} \cdot \frac{x \log a}{3} \ldots\ldots \frac{x \log a}{r+1} \cdot a^{\theta x}$$

will evidently approach without limit to zero, when r becomes indefinitely great, whatever be the value of x. Hence we are at liberty to put

$$a^x = 1 + \frac{x}{1} \log a + \frac{x^2}{1.2} (\log a)^2 + \frac{x^3}{1.2.3} (\log a)^3 + \ldots\ldots,$$

the series being supposed to be carried on to infinity.

Ex. 2. Let $f(x) = \log (x + 1)$:

then

$$f'(x) = (x+1)^{-1}, f''(x) = -(x+1)^{-2}, f'''(x) = (-)^2 . 1.2 (x+1)^{-3}, \ldots$$

and generally $f^r(x) = (-)^{r+1}. 1.2.3\ldots.(r-1) . (x+1)^{-r}$;

whence

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = (-)^2 . 1.2, \ldots,$$
$$f^r(0) = (-)^{r+1}. 1.2.3\ldots.(r-1).$$

Hence $$\log (x+1) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + (-)^{r+1} \frac{x^r}{r}$$
$$+ (-)^{r+2}. \frac{(1+\theta x)^{-r-1}}{r+1} . x^{r+1}.$$

If x be positive, then the expression R'_r will diminish without limit with the increase of r, provided that

$$\frac{x}{1+\theta x} < 1, \quad x < 1 + \theta x, \quad x < \frac{1}{1-\theta}:$$

but 1 is evidently the least value of $\frac{1}{1-\theta}$: hence, if x have any positive value between 0 and 1,

$$\log (x+1) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots,$$

the series being prolonged to infinity.

If x be negative, let $-z$ represent its value: then

$$R' = (-)^{r+2} \frac{1}{r+1} \left(\frac{-z}{1-\theta z}\right)^{r+1},$$

and, in order that we may be sure of the propriety of regarding the series for log $(x + 1)$ as infinite, we must have

$$\frac{z}{1-\theta z} < 1, \quad z < 1 - \theta z, \quad z < \frac{1}{1+\theta};$$

hence we are obliged to suppose that z is $< \frac{1}{2}$, or that the negative values of x are comprised between $-\frac{1}{2}$ and 0.

If however we take the other formula for R'_r, we see that

$$\begin{aligned} R'_r &= (-)^{r+2}(1-\theta_1)^r . (1+\theta_1 x)^{-r-1} x^{r+1} \\ &= (-)^{r+2}(1-\theta_1)^{-1} . \left(\frac{x-\theta_1 x}{1+\theta_1 x}\right)^{r+1} \\ &= (-)^{r+2}(1-\theta_1)^{-1} . \left(\frac{-z+\theta_1 z}{1-\theta_1 z}\right)^{r+1}, \end{aligned}$$

if we put $x = -z$. Then the expression

$$\frac{z-\theta_1 z}{1-\theta_1 z}$$

will be less than unity, provided that

$$z - \theta_1 z < 1 - \theta_1 z, \quad \text{or } z < 1.$$

Thus we see that the latter formula for R'_r allows a greater range of negative values for x than we could have inferred from the former.

If then x have any value comprised between the limits -1 and $+1$, we know that

$$\log(x+1) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots,$$

supposing this series to be continued to infinity.

Ex. 3. Let $\qquad f(x) = \sin x.$

Then $f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \ldots$

whence $f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \ldots$

and therefore

$$f(x) = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots\ldots + R_r',$$

where $$R_r' = \frac{x^{r+1}}{1.2.3.\dots.(r+1)} \, . \, (-)^{\frac{r+1}{2}} \sin(\theta x)$$

or $$R_r' = \frac{x^{r+1}}{1.2.3.\dots.(r+1)} \, . \, (-)^{\frac{r-1}{2}} \cos(\theta x),$$

accordingly as r is odd or even.

From these expressions it is plain that, when $r = \infty$,

$$R' = 0,$$

whatever be the value of x.

Hence, generally, whatever be the value of x,

$$\sin x = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots ,$$

the series being regarded as infinite.

We might prove in the same way that, whatever x be,

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots ,$$

a series likewise infinite.

Ex. 4. Let $$f(x) = (a+x)^m.$$

Then $f'(x) = m(a+x)^{m-1}$, $f''(x) = m(m-1)(a+x)^{m-3}$,

$$f'''(x) = m(m-1)(m-2)(a+x)^{m-3}, \dots$$

$$f^r(x) = m(m-1)(m-2)\dots(m-r+1).(a+x)^{m-r}:$$

whence $f'(0) = a^m, f'(0) = m\,a^{m-1}$, $f''(0) = m(m-1)\,a^{m-2}$,

$$f'''(0) = m(m-1)(m-2).a^{m-3}, \dots$$

$$f^r(0) = m(m-1)(m-2)\dots(m-r+1).\,a^{m-r}.$$

We have therefore

$$(a+x)^m = a^m + \frac{m}{1}a^{m-1}.x + \frac{m(m-1)}{1.2}a^{m-2}.x^2$$

$$+ \frac{m(m-1)(m-2)}{1.2.3}a^{m-3}.x^3 + \dots$$

$$+ \frac{m(m-1)(m-2)\dots(m-r+1)}{1.2.3\dots r}a^{m-r}.x^r + R_r'.$$

where $$R_r' = \frac{m(m-1)(m-2)\dots\dots(m-r)}{1.2.3\dots\dots(r+1)} \cdot (a+\theta x)^{m-r-1} . x^{r+1}$$

$$= \frac{m(m-1)(m-2)\dots\dots(m-r)}{1.2.3\dots\dots(r+1)} \left(\frac{x}{a+\theta x}\right)^{r+1} . (a+\theta x)^{m}.$$

Suppose that $x =$ a positive quantity: then, in order that R_r' may become zero, when r is made indefinitely great, since, as may easily be seen,

$$\frac{m(m-1)(m-2)\dots\dots(m-r)}{1.2.3\dots\ (r+1)}$$

is not infinite, it is sufficient that

$$\frac{x}{a+\theta x} < 1, \quad x < a + \theta x, \quad x < \frac{a}{1-\theta},$$

and therefore, *a fortiori*, that

$$x < a.$$

If x be negative, it will be convenient to have recourse to the second formula of Art. (90) for R_r', which gives

$$R_r' = \frac{m(m-1)(m-2)\dots\dots(m-r)}{1.2.3\dots\dots r} (1-\theta_1)^r . (a+\theta_1 x)^{m-r-1} . x^{r+1},$$

or, putting $-z$ for x,

$$R_r' = \frac{m(m-1)(m-2)\dots\dots(m-r)}{1.2.3\dots\dots r} . (1-\theta_1)^{-1} . \left\{(1-\theta_1) . \frac{-z}{a-\theta_1 z}\right\}^{r+1} . (a-\theta_1 z)^m.$$

Hence R_r' will be reduced to zero, when r becomes indefinitely great, if

$$\frac{z(1-\theta_1)}{a-\theta_1 z} < 1, \quad z(1-\theta_1) < a - \theta_1 z, \quad z < a.$$

Hence, for all values of x comprised between $-a$ and $+a$,

$$(a+x)^m = a^m + \frac{m}{1} a^{m-1} x + \frac{m(m-1)}{1.2} a^{m-2} x^2 + \frac{m(m-1)(m-2)}{1.2.3} a^{m-3} x^3 + \dots\dots \text{ ad infinitum}.$$

Ex. 5. Let $$f(x) = \Psi(x) + \varepsilon^{-\frac{1}{x^2}};$$

and suppose that $\Psi(x)$ is a function the development of which, by Stirling's theorem, may be continued indefinitely.

Then $f'(x) = \Psi'(x) + \frac{2}{x^3} \cdot \varepsilon^{-\frac{1}{x^2}}$, $f'(0) = \Psi'(0)$,

$$f''(x) = \Psi''(x) + \frac{2}{x^4}\left(\frac{2}{x^2} - 3\right) \cdot \varepsilon^{-\frac{1}{x^2}}, \quad f''(0) = \Psi''(0),$$

. . . .

the derivatives of $\varepsilon^{-\frac{1}{x^2}}$ being all zero when $x = 0$, as will be evident when we consider that

$$\frac{\frac{1}{x^m}}{a^{\frac{1}{x^n}}}, \quad \text{when } x = 0,$$

or

$$\frac{z^m}{a^{z^n}}, \quad \text{when } z = \infty,$$

is equal to zero for all positive values of m and n, a being any positive number greater than 1. Thus we see that in this example Stirling's series for the development of $f(x)$, when continued indefinitely, is convergent, and yet that it does not give for $f(x)$ a true value. In fact it makes the development of $f(x)$ the same as that of $\Psi(x)$, from which it would follow that $\varepsilon^{-\frac{1}{x^2}} = 0$ for all values of x. This shews that the mere convergency of the series, although necessary, is not sufficient for its truth, there being an additional condition, viz. the convergency of the remainder R_r' to zero.

Extension of Taylor's Theorem to Functions of two Variables.

92. Suppose that in the function $f(x, y)$, x and y are replaced by $x + h$ and $y + k$; our object is to obtain a development of the function

$$f(x + h,\ y + k)$$

by ascending powers of the increments h and k.

Putting $h = \alpha h'$, $k = \alpha k'$, we have

$$f(x + h,\ y + k) = f(x + \alpha h',\ y + \alpha k'),$$

which is a certain function of α, which we will denote by $\phi(\alpha)$.

By Stirling's theorem,

$$\phi(a) = \phi(0) + \frac{a}{1} \cdot \phi'(0) + \frac{a^2}{1.2}\phi''(0) + \frac{a^3}{1.2.3}\phi'''(0) + \ldots.(1).$$

Now

$$\phi'(a) = \frac{d}{dx}f(x + ah', y + ak') \cdot h' + \frac{d}{dy}f(x + ah', y + ak') \cdot k',$$

or, to adopt a more concise notation,

$$\phi'(a) = h'\frac{df_1}{dx} + k'\frac{df_1}{dy}:$$

differentiating again,

$$\phi''(a) = h'^2\frac{d^2f_1}{dx^2} + 2h'k'\frac{d^2f_1}{dxdy} + k'^2\frac{d^2f_1}{dy^2};$$

for a third differentiation

$$\phi'''(a) = h'^3\frac{d^3f_1}{dx^3} + 3h'^2k'\frac{d^3f_1}{dx^2dy} + 3h'k'^2\frac{d^3f_1}{dxdy^2} + k'^3\frac{d^3f_1}{dy^3},$$

and so on indefinitely, the law of derivation being obviously in accordance with the binomial theorem: we thus have, generally,

$$\phi^n(a) = h'^n\frac{d^nf_1}{dx^n} + \frac{n}{1}h'^{n-1}k'\frac{d^nf_1}{dx^{n-1}dy} + \frac{n(n-1)}{1.2}h'^{n-2}k'^2\frac{d^nf_1}{dx^{n-2}dy^2} + \ldots + k'^n\frac{d^nf_1}{dy^n}.$$

From the expressions for $\phi(a)$ and $\phi^n(a)$ it is plain that $\phi(0)=f$, where f is used to represent $f(x, y)$, and

$$\phi^n(0) = h'^n\frac{d^nf}{dx^n} + \frac{n}{1}h'^{n-1}k'\frac{d^nf}{dx^{n-1}dy} + \frac{n(n-1)}{1.2}h'^{n-2}k'^2\frac{d^nf}{dx^{n-2}dy^2} + \ldots + k'^n\frac{d^nf}{dy^n}.$$

Hence, from (1), substituting for $\phi(0)$, $\phi'(0)$, $\phi''(0)$, ... their values, and replacing ah', ak', by h, k, respectively, we have

$$\begin{aligned} f(x + h, y + k) & \\ = f + h\frac{df}{dx} + k\frac{df}{dy} & \\ + \frac{1}{1.2}\left(h^2\frac{d^2f}{dx^2} + 2hk\frac{d^2f}{dxdy} + k^2\frac{d^2f}{dy^2}\right) & \\ + \frac{1}{1.2.3}\left(h^3\frac{d^3f}{dx^3} + 3h^2k\frac{d^3f}{dx^2dy} + 3hk^2\frac{d^3f}{dxdy^2} + k^3\frac{d^3f}{dy^3}\right) & \\ + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2), & \end{aligned}$$

which is the required formula.

The formula (2) may be expressed more briefly by the aid of the separation of the symbols of differentiation from those of the function upon which they operate. Thus

$$h\frac{df}{dx} + k\frac{df}{dy} = \left(h\frac{d}{dx} + k\frac{d}{dy}\right)f,$$

$$k^2\frac{d^2f}{dx^2} + 2hk\frac{d^2f}{dx\,dy} + k^2\frac{d^2f}{dy^2} = \left(h\frac{d}{dx} + k\frac{d}{dy}\right)^2 f,$$

. . . .

the propriety of these symbolical expressions depending upon the principles of Art. (48), which shew that the laws of the combination of $\frac{d}{dx}$ and $\frac{d}{dy}$ are the same *inter se* as those of two symbols of magnitude.

We accordingly see that

$$f(x+h,\ y+k) = f + \left(h\frac{d}{dx} + k\frac{d}{dy}\right)f$$
$$+ \frac{1}{1.2}\left(h\frac{d}{dx} + k\frac{d}{dy}\right)^2 f + \frac{1}{1.2.3}\left(h\frac{d}{dx} + k\frac{d}{dy}\right)^3 f + \&c.$$
$$= \varepsilon^{h\frac{d}{dx} + k\frac{d}{dy}} f \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Since h and k are any quantities whatever, we may put $h = dx$, $k = dy$, in which case

$$h\frac{d}{dx} = \frac{hd_x}{dx} = d_x, \quad k\frac{d}{dy} = \frac{kd_y}{dy} = d_y;$$

whence there is

$$f(x+dx,\ y+dy) = \varepsilon^{d_x + d_y}.f \ldots\ldots\ldots\ldots (4);$$

or, since $d_x + d_y = D$, D denoting total differentiation,

$$f(x+dx,\ y+dy) = \varepsilon^{D}.f \ \ldots\ldots\ldots\ldots (5).$$

Cor. The method of development which we have applied to a function of two variables may obviously be extended to a function of any number of variables whatever. Thus

$$f(x+h,\ y+k,\ z+l, \ldots) = \varepsilon^{h\frac{d}{dx} + k\frac{d}{dy} + l\frac{d}{dz} + \ldots}.f,$$

whence also

$$f(x+dx,\ y+dy,\ z+dz, \ldots) = \varepsilon^{dx + dy + d_z + \ldots}.f.$$

Failure of the Development of f(x + h, y + k) *by Taylor's Theorem.*

93. The development of $f(x + h, y + k)$, given in the preceding article, fails for particular values of x and y, whenever f or any of its partial differential coefficients becomes infinite; this failure being consequent upon the failure of Stirling's theorem applied to the expansion of $\phi(\alpha)$. It will likewise cease to be applicable for particular values of x and y, which render f or its partial differential coefficients essentially indeterminate.

Limits and Remainders of the Development of f(x + h, y + k).

94. Let R_r' be the value of the remainder which must complete the series (1) of Art. 92, supposing this series to be stopped at the end of the $(r + 1)^{\text{th}}$ term; then, by Art. (90),

$$R_r' = \frac{\alpha^{r+1}}{1.2.3 \dots (r+1)} \cdot \phi^{r+1}(\theta\alpha) = \frac{(1-\theta_1)^r \alpha^{r+1}}{1.2.3 \dots r} \cdot \phi^{r+1}(\theta_1\alpha),$$

θ, θ_1, denoting certain unknown numbers comprised between 0 and 1. Hence, if we stop at the term

$$\left(h\frac{d}{dx} + k\frac{d}{dy}\right)^r \frac{f}{1.2.3 \dots r}$$

of the series for the development of $f(x + h, y + k)$, we must add the complementary remainder ρ_r, where

$$\rho_r = \left(h\frac{d}{dx} + k\frac{d}{dy}\right)^{r+1} . \frac{f(x + \theta h,\ y + \theta k)}{1.2.3 \dots (r+1)},$$

or
$$= (1-\theta_1)^r . \left(h\frac{d}{dx} + k\frac{d}{dy}\right)^{r+1} . \frac{f(x + \theta_1 h,\ y + \theta_1 k)}{1.2.3 \dots r}.$$

The numerical fractions θ, θ_1, which enter into these formulæ, being unknown, we cannot employ the formulæ for the actual computation of ρ_r: they serve only in fact to determine limits between which the value of ρ_r must lie.

The value of R_r' is equal to the definite integral

$$\frac{1}{1.2.3\ldots r}\int_0^{\alpha} \phi^{r+1}(\alpha-\beta)\,\beta^r d\beta,$$

whence, for the actual determination of ρ_r, we have

$$\rho_r=\frac{1}{1.2.3\ldots r}\int_0^{\alpha}\left(h'\frac{d}{dx}+k'\frac{d}{dy}\right)^{r+1} f\{x+(\alpha-\beta)h',\ y+(\alpha-\beta)k'\}.\beta^r d\beta$$

$$=\frac{1}{1.2.3\ldots r}\cdot\frac{1}{\alpha^{r+1}}\int_0^{\alpha}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^{r+1}.f\{x+h-h'\beta,\ y+k-k'\beta\}.\beta^r d\beta,$$

or, putting $\beta=\dfrac{\beta'\alpha}{h}$,

$$\rho_r=\frac{1}{1.2.3\ldots r}\int_0^{h}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^{r+1}.f\left(x+h-\beta',\ y+k-\frac{k\beta'}{h}\right).\frac{\beta'^r d\beta'}{h^{r+1}}$$

$$=\frac{1}{1.2.3\ldots r}\cdot\frac{1}{h^{r+1}}\int_0^{h}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^{r+1}.f\left(x+h-\beta,\ y+k-\frac{k\beta}{h}\right).\beta^r d\beta.$$

By analogous reasoning we might shew also, as an equivalent formula, that

$$\rho_r=\frac{1}{1.2.3\ldots r}\cdot\frac{1}{k^{r+1}}\int_0^{k}\left(h\frac{d}{dx}+k\frac{d}{dy}\right)^{r+1}.f\left(x+h-\frac{h\beta}{k},\ y+k-\beta\right).\beta^r d\beta.$$

In precisely the same way we might investigate symbolical formulæ for the remainder in the development of a function $f(x+h,\ y+k,\ z+l,\ \ldots)$, $x, y, z, \ldots$ being any number of variables.

Example of the Application of Taylor's Theorem for two Variables.

95. Let $f(x, y)=0$ be the equation to a curve, $f(x, y)$ being a rational function of x and y; and let it be proposed to transform this equation into an equivalent one for a new origin (α, β).

Putting $\alpha+x$, $\beta+y$, for x, y, we have, for the transformed equation,

$$0=f(\alpha+x,\ \beta+y),$$

or, by Taylor's theorem, the dimensions of the proposed equation in x and y being n,

$$0 = f + x\frac{df}{d\alpha} + y\frac{df}{d\beta} + \frac{1}{1.2}\left(x^2\frac{d^2f}{d\alpha^2} + 2xy\frac{d^2f}{d\alpha\,d\beta} + y^2\frac{d^2f}{d\beta^2}\right) + \dots$$

$$+ \frac{1}{1.2.3\dots n}\left(x^n\frac{d^nf}{d\alpha^n} + \frac{n}{1}x^{n-1}y\frac{d^nf}{d\alpha^{n-1}d\beta} + \frac{n(n-1)}{1.2}x^{n-2}y^2\frac{d^nf}{d\alpha^{n-2}d\beta^2}\right.$$

$$\left. + \dots + y^n\frac{d^nf}{d\beta^n}\right).$$

Let, for instance, the equation be

$$0 = Ax^2 + By^2 + 2Cxy + 2A'x + 2B'y + C':$$

then $f = A\alpha^2 + B\beta^2 + 2C\alpha\beta + 2A'\alpha + 2B'\beta + C'$,

$$\frac{df}{d\alpha} = 2A\alpha + 2C\beta + 2A',$$

$$\frac{df}{d\beta} = 2B\beta + 2C\alpha + 2B',$$

$$\frac{d^2f}{d\alpha^2} = 2A, \quad \frac{d^2f}{d\alpha\,d\beta} = 2C, \quad \frac{d^2f}{d\beta^2} = 2B:$$

the transformed equation will therefore be

$$0 = A\alpha^2 + B\beta^2 + 2C\alpha\beta + 2A'\alpha + 2B'\beta + C'$$
$$+ 2x(A\alpha + C\beta + A') + 2y(B\beta + C\alpha + B')$$
$$+ Ax^2 + By^2 + 2Cxy.$$

Stirling's Theorem applied to Functions of two Variables.

96. If, in the development of $f(x + h, y + k)$ by Taylor's theorem, we substitute 0 and 0′ in place of x and y, where 0 and 0′ are used to denote zero values respectively of x and y, and then replace h, k, by x, y, respectively, we have

$$f(x, y) = f(0, 0') + \left(x\frac{d}{d0} + y\frac{d}{d0'}\right)f(0, 0')$$
$$+ \frac{1}{1.2}\left(x\frac{d}{d0} + y\frac{d}{d0'}\right)^2 f(0, 0')$$
$$+ \frac{1}{1.2.3}\left(x\frac{d}{d0} + y\frac{d}{d0'}\right)^3 f(0, 0')$$
$$+ \&c.,$$

which constitutes an extension of Stirling's theorem to functions of two variables.

The expressions for the limits and remainder may be obtained at once from those for the development of $f(x+h,\ y+k)$, by first putting $x=0,\ y=0'$, and then replacing $h,\ k$, respectively by $x,\ y$.

Lagrange's Formula for the Development of Implicit Functions.

97. Suppose that

$$u = f(y) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

y being an implicit function of x and z by virtue of the equation

$$y = z + x\,\phi(y) \ldots\ldots\ldots\ldots\ldots (2).$$

The object of Lagrange's formula, which we proceed to investigate, is to enable us to develop u in a series arranged by ascending powers of x, and which does not involve y.

If $\Psi(y)$ be any function of y, y being a function of x and z, then

$$\frac{d}{dx}\left\{\Psi(y)\,.\,\frac{dy}{dz}\right\} = \frac{d}{dz}\left\{\Psi(y)\,.\,\frac{dy}{dx}\right\} \ldots\ldots (3);$$

for it is plain that each of these expressions is equal to

$$\Psi'(y)\,.\,\frac{dy}{dx}\,.\,\frac{dy}{dz} + \Psi(y)\,.\,\frac{d^2y}{dx\,dz}\,.$$

Differentiating (2), considering z constant, we have

$$\frac{dy}{dx} = \phi(y) + x\,\phi'(y)\,.\,\frac{dy}{dx},$$

$$\{1 - x\,\phi'(y)\}\,\frac{dy}{dx} = \phi(y) \ldots\ldots\ldots (4);$$

and, considering x constant,

$$\frac{dy}{dz} = 1 + x\,\phi'(y)\,.\,\frac{dy}{dz},$$

$$\{1 - x\,\phi'(y)\}\,\frac{dy}{dz} = 1 \ldots\ldots\ldots\ldots (5).$$

From (4) and (5) there is

$$\frac{dy}{dx} = \phi(y) \cdot \frac{dy}{dz} \quad \ldots\ldots\ldots\ldots \quad (6).$$

From (1), differentiating on the supposition that z is constant,

$$\frac{du}{dx} = f'(y) \cdot \frac{dy}{dx}$$

$$= f'(y) \cdot \phi(y) \cdot \frac{dy}{dz}, \text{ from (6).}$$

Hence, $f'(y) \cdot \phi(y)$ taking the place of $\Psi(y)$ in (3), we have

$$\frac{d^2u}{dx^2} = \frac{d}{dz}\left\{f'(y) \cdot \phi(y) \cdot \frac{dy}{dx}\right\}$$

$$= \frac{d}{dz}\left[f'(y) \cdot \{\phi(y)\}^2 \cdot \frac{dy}{dz}\right].$$

Differentiating again, we have

$$\frac{d^3u}{dx^3} = \frac{d^2}{dz\,dx}\left[f'(y) \cdot \{\phi(y)\}^2 \cdot \frac{dy}{dz}\right]$$

$$= \frac{d^2}{dz^2}\left[f'(y) \cdot \{\phi(y)\}^2 \cdot \frac{dy}{dx}\right], \text{ by (3),}$$

$$= \frac{d^2}{dz^2}\left[f'(y) \cdot \{\phi(y)\}^3 \cdot \frac{dy}{dz}\right], \text{ by (6).}$$

Proceeding in the same way it is evident that we shall have, generally,

$$\frac{d^nu}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left[f'(y) \cdot \{\phi(y)\}^n \cdot \frac{dy}{dz}\right].$$

By Stirling's theorem,

$$u = (u)_{x=0} + \frac{x}{1}\left(\frac{du}{dx}\right)_{x=0} + \frac{x^2}{1.2}\left(\frac{d^2u}{dx^2}\right)_{x=0} + \frac{x^3}{1.2.3}\left(\frac{d^3u}{dx^3}\right)_{x=0} + \ldots.$$

But from (2) and (5) it appears that $y = z$, and $\frac{dy}{dz} = 1$, when $x = 0$.

Hence $$\left(\frac{d^n u}{dx^n}\right)_{x=0} = \frac{d^{n-1}}{dz^{n-1}}\left[f'(z)\,.\,\{\phi(z)\}^n\right],$$

$$u = f(z) + \frac{x}{1}\,.\,[f'(z)\,.\,\phi(z)] + \frac{x^2}{1.2}\,.\,\frac{d}{dz}\left[f'(z)\,.\,\{\phi(z)\}^2\right]$$
$$+ \frac{x^3}{1.2.3}\,\frac{d^2}{dz^2}\left[f'(z).\{\phi(z)\}^3\right] + \frac{x^4}{1.2.3.4}\,\frac{d^3}{dz^3}\left[f'(z).\{\phi(z)\}^4\right] + \ldots (7).$$

In the particular case, when

$$u = z + x\,\phi(u),$$

or, when $f(y) = y$, we have $f'(z) = 1$, and, instead of the formula (7), we have

$$u = z + \frac{x}{1}\,.\,[\phi\,(z)] + \frac{x^2}{1.2}\,\frac{d}{dz}\left[\{\phi(z)\}^2\right] + \frac{x^3}{1.2.3}\,\frac{d^2}{dz^2}\,.\,\left[\{\phi(z)\}^3\right]$$
$$+ \frac{x^4}{1.2.3.4}\,\frac{d^3}{dz^3}\left[\{\phi(z)\}^4\right] + \ldots\ldots \quad (8).$$

This theorem was given by Lagrange, in the *Mémoires de Berlin* for the year 1770, (see also his *Equations Numeriques*, Note XI.), as a generalization of a particular development obtained by Lambert for the expression of the roots of certain algebraic equations: Lambert's results were published in the year 1758. A demonstration of Lagrange's theorem, due to Cauchy, which involves some important reflections respecting the convergency of the series, may be seen in Moigno's *Leçons de Calcul Différentiel et de Calcul Intégral*, tom. I., pp. 162–172. An expression for the limits of error committed in stopping at any term of the series, has been given by Murphy in the fourth volume of the *Cambridge Philosophical Transactions*.

Ex. 1. Let $$u = z + x \sin u.$$

The determination of u in terms of x and z is a celebrated problem in astronomy called Kepler's problem: the variable z denotes a quantity which varies as the time, the coefficient x represents the eccentricity of the elliptical orbit of a planet, and the variable u is an angle called the *Eccentric Anomaly*.

$$\phi(z) = \sin z,$$

$$\frac{d}{dz}[\{\phi(z)\}^2] = \frac{d}{dz}\{(\sin z)^2\} = 2 \sin z \cos z = \sin 2z,$$

$$\frac{d^2}{dz^2}\{\phi(z)\}^3 = \frac{d^2}{dz^2}.(\sin z)^3 = \frac{d^2}{dz^2}.(\tfrac{3}{4}\sin z - \tfrac{1}{4}\sin 3z) = -\tfrac{3}{4}\sin z + \tfrac{9}{4}\sin 3z.$$

Hence, as far as the term involving x^3, we have, by formula (8),

$$u = z + \sin z\,.\,\frac{x}{1} + \sin 2z\,.\,\frac{x^2}{1.2} + \tfrac{3}{4}(3 \sin 3z - \sin z)\,.\,\frac{x^3}{1.2.3} + \ldots$$

Ex. 2. Let $y = z + x \sin y,$ and $u = \sin y.$

Then $f(z) = \sin z,\ \ \phi(z) = \sin z,\ \ f'(z) = \cos z:$

$$f'(z)\,.\,\phi(z) = \sin z \cos z = \tfrac{1}{2}\sin 2z,$$

$$\begin{aligned} f'(z)\,.\,\{\phi(z)\}^2 &= \cos z\,.\,\sin^2 z \\ &= \tfrac{1}{2}\sin 2z\,.\,\sin z \\ &= \tfrac{1}{4}(\cos z - \cos 3z), \end{aligned}$$

$$\frac{d}{dz}[f'(z)\,.\,\{\phi(z)\}^2] = \tfrac{1}{4}(3 \sin 3z - \sin z),$$

$$\begin{aligned} f'(z)\,.\,\{\phi(z)\}^3 &= \cos z\,.\,\sin^3 z \\ &= \tfrac{1}{4}\sin 2z\,.\,(1 - \cos 2z) \\ &= \tfrac{1}{8}(2 \sin 2z - \sin 4z), \end{aligned}$$

$$\frac{d}{dz}[f'(z)\,.\,\{\phi(z)\}^3] = \tfrac{1}{2}(\cos 2z - \cos 4z),$$

$$\frac{d^2}{dz^2}[f'(z)\,.\,\{\phi(z)\}^3] = (2 \sin 4z - \sin 2z).$$

Hence, as far as the term involving x^3, we have by formula (7),

$$u = \sin z + \tfrac{1}{2}\sin 2z\,.\,\frac{x}{1} + \tfrac{1}{4}(3 \sin 3z - \sin z)\,.\,\frac{x^2}{1.2}$$
$$+ (2 \sin 4z - \sin 2z)\,.\,\frac{x^3}{1.2.3} + \ldots.$$

Laplace's Formula for the Development of Implicit Functions.

98. Suppose that $u = f(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (1),

and $y = F\{z + x\phi(y)\} \ldots\ldots\ldots\ldots$ (2).

Differentiating (2), considering z constant, we have

$$\frac{dy}{dx} = F'\{z + x\phi(y)\}.\left\{\phi(y) + x\phi'(y).\frac{dy}{dx}\right\},$$

$$[1 - F'\{z + x\phi(y)\}.x\phi'(y)]\frac{dy}{dx} = F'\{z + x\phi(y)\}.\phi(y)\ldots(3);$$

and, considering x constant,

$$\frac{dy}{dz} = F'\{z + x\phi(y)\}.\left\{1 + x\phi'(y).\frac{dy}{dz}\right\},$$

$$[1 - F'\{z + x\phi(y)\}.x\phi'(y)].\frac{dy}{dz} = F'\{z + x\phi(y)\}\ \ldots\ldots(4).$$

From (3) and (4) we see that

$$\frac{dy}{dx} = \phi(y).\frac{dy}{dz}\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

From (1), differentiating on the supposition that z is constant,

$$\frac{du}{dx} = f'(y).\frac{dy}{dx}$$

$$= f'(y).\phi(y).\frac{dy}{dz}, \text{ from (5).}$$

Differentiating again, and bearing in mind the theorem established in the preceding article, viz.

$$\frac{d}{dx}\left\{\Psi(y).\frac{dy}{dz}\right\} = \frac{d}{dz}\left\{\Psi(y).\frac{dy}{dx}\right\},$$

we have

$$\frac{d^2u}{dx^2} = \frac{d}{dz}\left\{f'(y).\phi(y).\frac{dy}{dx}\right\}$$

$$= \frac{d}{dz}\left[f'(y).\{\phi(y)\}^2.\frac{dy}{dz}\right].$$

Differentiating again, we have

$$\frac{d^3u}{dx^3} = \frac{d^2}{dzdx}\left[f'(y).\{\phi(y)\}^2\frac{dy}{dz}\right]$$

$$= \frac{d^2}{dz^2}\left[f'(y).\{\phi(y)\}^2\frac{dy}{dx}\right]$$

$$= \frac{d^2}{dz^2}\left[f'(y).\{\phi(y)\}^3\frac{dy}{dz}\right].$$

Proceeding in the same way we have, generally,

$$\frac{d^n u}{dx^n} = \frac{d^{n-1}}{dz^{n-1}} \left[f'(y) \cdot \{\phi(y)\}^n \frac{dy}{dz}\right].$$

From (2) and (4) it appears that, $y = F(z)$, $\frac{dy}{dz} = F'(z)$, when $x = 0$, and therefore

$$\left(\frac{d^n u}{dx^n}\right)_{x=0} = \frac{d^{n-1}}{dz^{n-1}} \left[f'\{F(z)\} \cdot [\phi \cdot \{F(z)\}]^n \cdot F'(z)\right].$$

Hence, by Stirling's theorem, viz.

$$u = (u)_{x=0} + \frac{x}{1}\left(\frac{du}{dx}\right)_{x=0} + \frac{x^2}{1.2}\left(\frac{d^2u}{dx^2}\right)_{x=0} + \frac{x^3}{1.2.3}\left(\frac{d^3u}{dx^3}\right)_{x=0} + \dots,$$

we have, putting for the sake of brevity $f\{F(z)\} = f_1(z)$, and $\phi\{F(z)\} = \phi_1(z)$,

$$u = f_1(z) + \frac{x}{1} \cdot [f_1'(z) \cdot \phi_1(z)] + \frac{x^2}{1.2} \frac{d}{dz} [f_1'(z) \cdot \{\phi_1(z)\}^2]$$
$$+ \frac{x^3}{1.2.3} \frac{d^2}{dz^2} [f_1'(z) \cdot \{\phi_1(z)\}^3] + \dots\dots,$$

which is Laplace's formula, first given by him in the *Mémoires de l'Académie des Sciences*, 1777, p. 99. Lagrange's formula is evidently a particular case of Laplace's, from which it is at once derived by putting $F(z) = z$.

DIFFERENTIAL CALCULUS.

SECOND PART.

GEOMETRICAL APPLICATIONS.

CHAPTER I.

TANGENCY.

Definition of a Tangent and of a Normal.

99. Let P, Q, be two points of a curve AB, (fig. 1), and suppose that an indefinite straight line $H'K'$ is drawn through these two points. Conceive the point Q to move towards P; then the secant $H'K'$ will keep tending towards a certain limiting position, and ultimately, that is, just as Q is on the point of coalescing with P, is said to be a *tangent to the curve at P*. An

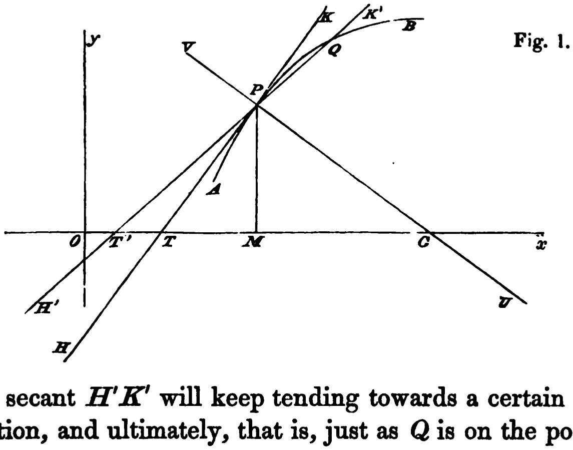


Fig. 1.

indefinite straight line UV drawn through P, at right angles to HK, the limiting position of $H'K'$, that is, at right angles to the tangent at P, is called a *normal to the curve at the point P.*

Inclinations of the Tangent and the Normal at any point of a Curve to the coordinate Axes.

100. Let Ox, Oy, be rectangular axes of coordinates, (fig. 1), AB being a curve contained in the plane xOy. Let x, y, be the coordinates OM, MP, of P, and x', y', the coordinates of Q. Let c denote the length of the chord PQ. Then the cosine, sine, and tangent of the angle $K'T'x$, T' being the point in which $H'K'$ cuts Ox, are respectively

$$\frac{x'-x}{c}, \quad \frac{y'-y}{c}, \quad \frac{y'-y}{x'-x}.$$

When x' approaches indefinitely near to x, in consequence of the approach of Q towards P, then ultimately, α denoting the angle KTx, where T is the point in which HK cuts Ox, s representing the arc AP, and s' the arc AQ,

$$\cos\alpha = \text{the limit of } \frac{x'-x}{c} = \text{the limit of } \frac{x'-x}{s'-s}\cdot\frac{s'-s}{c},$$

$$\sin\alpha = \text{the limit of } \frac{y'-y}{c} = \text{the limit of } \frac{y'-y}{s'-s}\cdot\frac{s'-s}{c},$$

$$\tan\alpha = \text{the limit of } \frac{y'-y}{x'-x}.$$

But, by Newton's Seventh Lemma, in the first section of the Principia, we know that

$$\text{the limit of } \frac{s'-s}{c} = 1:$$

hence $$\cos\alpha = \frac{dx}{ds}, \quad \sin\alpha = \frac{dy}{ds}, \quad \tan\alpha = \frac{dy}{dx}.$$

Again, β denoting the angle PGO, G being the point in which the normal UV cuts the axis of x,

$$\cos\beta = \sin\alpha = \frac{dy}{ds}, \quad \sin\beta = \cos\alpha = \frac{dx}{ds}, \quad \tan\beta = \cot\alpha = \frac{dx}{dy}.$$

We may express cos α and sin α in terms of dx and dy alone, without ds: thus, adding together the squares of the two equations

$$\cos\alpha = \frac{dx}{ds}, \quad \sin\alpha = \frac{dy}{ds},$$

we get
$$1 = \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2},$$

or
$$ds^2 = dx^2 + dy^2.$$

Hence $$\cos^2\alpha = \frac{dx^2}{dx^2 + dy^2}, \quad \sin^2\alpha = \frac{dy^2}{dx^2 + dy^2}.$$

Again, supposing the equation to the curve to be

$$u = 0,$$

we have, by differentiation,

$$\frac{du}{dx}\,dx + \frac{du}{dy}\,dy = 0,$$

and therefore
$$dx : dy :: -\frac{du}{dy} : \frac{du}{dx},$$

whence
$$\sin^2\beta = \cos^2\alpha = \frac{dx^2}{dx^2 + dy^2} = \frac{\frac{du^2}{dy^2}}{\frac{du^2}{dx^2} + \frac{du^2}{dy^2}},$$

$$\cos^2\beta = \sin^2\alpha = \frac{dy^2}{dx^2 + dy^2} = \frac{\frac{du^2}{dx^2}}{\frac{du^2}{dx^2} + \frac{du^2}{dy^2}},$$

$$\cot\beta = \tan\alpha = \frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}}.$$

Ex. To find the inclination of the tangent at any point of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

to the coordinate axes.

Here
$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

$$\frac{du}{dx} = \frac{2x}{a^2}, \quad \frac{du}{dy} = \frac{2y}{b^2}.$$

Hence
$$\cos^2 \alpha = \frac{\left(\frac{2y}{b^2}\right)^2}{\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2} = \frac{\frac{y^2}{b^4}}{\frac{x^2}{a^4} + \frac{y^2}{b^4}},$$

$$\sin^2 \alpha = \frac{\left(\frac{2x}{a^2}\right)^2}{\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2} = \frac{\frac{x^2}{a^4}}{\frac{x^2}{a^4} + \frac{y^2}{b^4}},$$

$$\tan \alpha = -\frac{\frac{2x}{a^2}}{\frac{2y}{b^2}} = -\frac{b^2 x}{a^2 y}.$$

The negative sign in the expression for tan α, supposing x and y to be positive, shews that the angle PTx is obtuse, instead of acute, as in fig. (1).

Equations to the Tangent and the Normal at any Point of a Curve.

101. Let R, fig. (2), be any point whatever in the tangent HK at the point P of the curve AB.

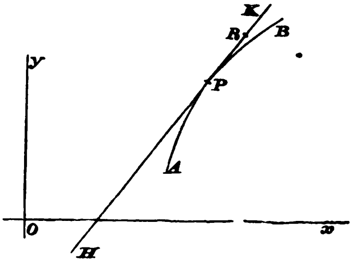


Fig. 2.

Let (x, y), (x', y'), be the coordinates of P, R, respectively. Then, r denoting the distance between P and R,

$$x' - x = r\cos\alpha = r\frac{dx}{ds},$$

$$y' - y = r\sin\alpha = r\frac{dy}{ds};$$

and therefore
$$\frac{x' - x}{dx} = \frac{y' - y}{dy},$$

a form of the equation to the tangent at *P*. An equation, usually more convenient, may be obtained in the following manner.

Differentiating the equation $u = 0$ to the curve, we have

$$\frac{du}{dx}dx + \frac{du}{dy}dy = 0:$$

multiplying the former and the latter term of this equation by the two equal quantities

$$\frac{x' - x}{dx},\ \frac{y' - y}{dy},$$

respectively, we obtain, for the equation to the tangent,

$$(x' - x)\frac{du}{dx} + (y' - y)\frac{du}{dy} = 0.$$

The equation to a line through the point x, y, at right angles to the tangent, that is, the equation to the normal at *P*, will therefore be, as we know by the condition of perpendicularity given in treatises on algebraic geometry,

$$(x' - x)\,dx + (y' - y)\,dy = 0,$$

or
$$\frac{x' - x}{\frac{du}{dx}} = \frac{y' - y}{\frac{du}{dy}}.$$

Ex. To find the equations to the tangent and normal at any point of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Putting for $\frac{du}{dx}$, $\frac{du}{dy}$, their values $\frac{2x}{a^2}$, $\frac{2y}{b^2}$, we get, for the equation to the tangent,

$$(x' - x)\frac{2x}{a^2} + (y' - y)\frac{2y}{b^2} = 0,$$

or, by virtue of the equation to the ellipse,

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

and, for the equation to the normal,

$$(x' - x)\frac{a^2}{x} = (y' - y)\frac{b^2}{y}.$$

Distance of the Origin of Coordinates from the Tangent.

102. Let δ denote the distance of the origin from the tangent, ε being the inclination of δ to the axis of x. Then the equation to the tangent must coincide with the equation

$$x' \cos \varepsilon + y' \sin \varepsilon = \delta.$$

Comparing this equation with each of the equations to the tangent, viz.

$$\frac{x' - x}{dx} = \frac{y' - y}{dy},$$

$$x'\frac{du}{dx} + y'\frac{du}{dy} = x\frac{du}{dx} + y\frac{du}{dy},$$

we see that

$$\frac{1}{dx\left(\frac{x}{dx} - \frac{y}{dy}\right)} = \frac{\cos \varepsilon}{\delta} = \frac{\frac{du}{dx}}{x\frac{du}{dx} + y\frac{du}{dy}},$$

$$\frac{1}{dy\left(\frac{y}{dy} - \frac{x}{dx}\right)} = \frac{\sin \varepsilon}{\delta} = \frac{\frac{du}{dy}}{x\frac{du}{dx} + y\frac{du}{dy}};$$

and consequently

$$\frac{dx^2 + dy^2}{(xdy - ydx)^2} = \frac{1}{\delta^2} = \frac{\dfrac{du^2}{dx^2} + \dfrac{du^2}{dy^2}}{\left(x\dfrac{du}{dx} + y\dfrac{du}{dy}\right)^2},$$

which determines the value of δ.

Intercepts of the Tangent.

103. Let x_0 be the value of x' in the equation to the tangent at any point of a curve when $y' = 0$. The length x_0, viz. OT in fig. (1), is called the *intercept* of the tangent on the axis of x.

It is easily seen from the two forms of the equation to the tangent that

$$\frac{x\,dy - y\,dx}{dy} = x_0 = \frac{x\dfrac{du}{dx} + y\dfrac{du}{dy}}{\dfrac{du}{dx}};$$

in the same way, y_0 denoting the *intercept* of the tangent on the axis of y,

$$\frac{y\,dx - x\,dy}{dx} = y_0 = \frac{x\dfrac{du}{dx} + y\dfrac{du}{dy}}{\dfrac{du}{dy}}.$$

Subtangent.

104. The portion MT, fig. (1), of the axis of x, contained between its intersections with the ordinate and the tangent at P, is ordinarily called the ***subtangent at the point P***. It is evident, from the equations to the tangent, that MT is equal to

$$y\frac{dx}{dy} = x - x_0 = -y\frac{\dfrac{du}{dy}}{\dfrac{du}{dx}}.$$

Length of the Tangent.

105. The word *tangent* is sometimes used to denote the finite line PT, fig. (1), included between the point P of contact of the indefinite tangent HK with the curve and the point T in which the indefinite tangent cuts the axis of x. In this sense of the word it is plain that the length of the tangent is equal to

$$\{y^2 + (x - x_0)^2\}^{\frac{1}{2}}$$

$$= \left(y^2 + y^2 \frac{dx^2}{dy^2}\right)^{\frac{1}{2}};$$

or

$$= \frac{y}{\frac{du}{dx}} \left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right).$$

Normal and Subnormal.

106. The finite lines PG, GM, fig. (1), G being the intersection of the indefinite normal UV and the axis of x, are frequently called the *normal* and *subnormal* respectively. It is plain that the subnormal is equal to the value of $x' - x$ deduced from either of the equations to the normal, when $y' = 0$; or

$$y \frac{dy}{dx} = MG = - y \frac{\frac{du}{dx}}{\frac{du}{dy}},$$

and therefore

$$\left(y^2 + y^2 \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = PG = \left\{y^2 + y^2 \frac{\frac{du^2}{dx^2}}{\frac{du^2}{dy^2}}\right\}^{\frac{1}{2}}.$$

Form of the Equation to the Tangent to Curves of which the equations involve only rational functions of x *and* y.

107. Let $u = 0$ be the equation to a curve; and suppose that

$$u = u_0 + u_1 + u_2 + u_3 + \dots + u_n,$$

u_r denoting a homogeneous function of x and y of r dimensions. Then the equation to the tangent at any point x, y, will be

$$x'\left(\frac{du_1}{dx}+\frac{du_2}{dx}+\frac{du_3}{dx}+\dots+\frac{du_n}{dx}\right)+y'\left(\frac{du_1}{dy}+\frac{du_2}{dy}+\frac{du_3}{dy}+\dots+\frac{du_n}{dy}\right)$$
$$+nu_0+(n-1)u_1+(n-2)u_2+(n-3)u_3+\dots\dots+2u_{n-2}+u_{n-1}=0.$$

Lemma. Let v be any rational function of x and y, of r dimensions: then
$$v=\Sigma\,(cx^\alpha y^\beta),$$
where c is a constant coefficient, and $\alpha+\beta=r$, the term $cx^\alpha y^\beta$ being a type of all the terms. Then
$$\frac{dv}{dx}=\Sigma\,(c\alpha x^{\alpha-1}y^\beta),$$
$$\frac{dv}{dy}=\Sigma\,(c\beta x^\alpha y^{\beta-1}),$$
and therefore
$$x\frac{dv}{dx}+y\frac{dv}{dy}=\Sigma\,(c\alpha x^\alpha y^\beta)+\Sigma\,(c\beta x^\alpha y^\beta)$$
$$=\Sigma\,\{c\,(\alpha+\beta)\,x^\alpha y^\beta\}$$
$$=r\,\Sigma\,(cx^\alpha y^\beta)=rv.$$

By virtue of this Lemma we see that
$$x\frac{du}{dx}+y\frac{du}{dy}=\qquad u_1+2u_2+3u_3+\dots+(n-1)\,u_{n-1}+nu_n:$$
but $\qquad 0=nu_0+nu_1+nu_2+nu_3+\dots+nu_{n-1}+nu_n;$

hence
$$x\frac{du}{dx}+y\frac{du}{dy}=-nu_0-(n-1)\,u_1-(n-2)\,u_2-(n-3)\,u_3-\dots-2u_{n-2}-u_{n-1},$$
and therefore, from the equation
$$x'\frac{du}{dx}+y'\frac{du}{dy}=x\frac{du}{dx}+y\frac{du}{dy},$$
we get
$$x'\left(\frac{du_1}{dx}+\frac{du_2}{dx}+\frac{du_3}{dx}+\dots\dots+\frac{du_n}{dx}\right)$$
$$+y'\left(\frac{du_1}{dy}+\frac{du_2}{dy}+\frac{du_3}{dy}+\dots+\frac{du_n}{dy}\right)$$
$$+nu_0+(n-1)\,u_1+(n-2)\,u_2+(n-3)\,u_3+\dots+2u_{n-2}+u_{n-1}=0.$$

Ex. To find the equation to the tangent at any point of the conic section

$$ax^2 + by^2 + 2cxy + 2a'x + 2b'y + c' = 0.$$

By the above formula we see at once that

$$x'.(a' + cy + ax) + y'.(b' + cx + by) + c' + a'x + b'y = 0.$$

Oblique Axes.

108. The forms of the equations to the tangent are not altered if we suppose the axes to be oblique instead of rectangular.

Let P, P_1, (fig. 3.) be the two points in which the secant $H'K'$ cuts the curve AB, referred to oblique axes Ox, Oy.

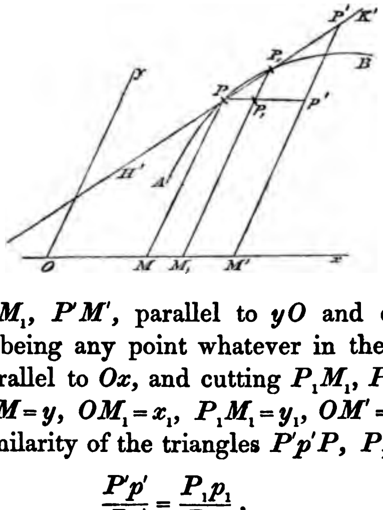


Fig. 3.

Draw PM, P_1M_1, $P'M'$, parallel to yO and cutting Ox in M, M_1, M', P' being any point whatever in the secant $H'K'$. Draw Pp_1p', parallel to Ox, and cutting P_1M_1, $P'M'$, in p_1, p'. Let $OM=x$, $PM=y$, $OM_1=x_1$, $P_1M_1=y_1$, $OM'=x'$, $P'M'=y'$. Then, by the similarity of the triangles $P'p'P$, P_1p_1P, we have

$$\frac{P'p'}{Pp'} = \frac{P_1p_1}{Pp_1},$$

or
$$\frac{y'-y}{x'-x} = \frac{y_1-y}{x_1-x}.$$

In the limit, when P_1 moves indefinitely near to P, x_1, y_1, become x, y, respectively, and

$$\frac{y_1-y}{x_1-x}$$

becomes $\frac{dy}{dx}$. The secant $H'K'$ also becomes a tangent to the curve at P. Thus the equation to the tangent at P is

$$\frac{y'-y}{x'-x}=\frac{dy}{dx},$$

or

$$\frac{x'-x}{dx}=\frac{y'-y}{dy};$$

whence also, as in Art. (101), we have, as another form of the equation,

$$(x'-x)\frac{du}{dx}+(y'-y)\frac{du}{dy}=0.$$

CHAPTER II.

ASYMPTOTES.

Definition of an Asymptote. Method of finding Asymptotes.

109. An asymptote is a tangent line to a curve, such that, although the distance between the origin and the point of contact is infinite, the perpendicular distance of the origin from the line is finite.

Ex. 1. Take for instance the curve

$$u = a^x - y = 0;$$

where a is supposed to be a number greater than unity: then, by the formula

$$\frac{du}{dx} x' + \frac{du}{dy} y' = \frac{du}{dx} x + \frac{du}{dy} y,$$

we have, for the equation to the tangent at any point,

$$\log a . a^x . x' - y' = \log a . a^x . x - y$$
$$= x \log a . a^x - a^x.$$

Suppose now that $x = -\infty$: then $a^x = 0$, and

$$x . a^x = -\frac{z}{a^z}, \quad z = \infty,$$
$$= -\frac{1}{\log a . a^z} = 0.$$

Thus the equation to the tangent becomes

$$y' = 0,$$

the equation to an asymptote coincident with the axis of x, the coordinates of the point of contact being $-\infty$, 0.

Ex. 2. Take the curve

$$u = ax^2 + x^3 - y^3 = 0.$$

Then the equation to the tangent will be

$$(2ax + 3x^2)\,x' - 3y^2y' = 2ax^2 + 3\,(x^3 - y^3) = -\,ax^2,$$

or

$$\left(\frac{2a}{x} + 3\right)x' - 3\,\frac{y^2}{x^2}\,y' = -\,a.$$

Suppose that $x = \infty$; then from the equation to the curve we have

$$\frac{a}{x} + 1 - \frac{y^3}{x^3} = 0, \qquad \frac{y}{x} = 1:$$

thus the equation to the tangent becomes

$$3x' - 3y' = -\,a,$$

which represents an asymptote.

From the preceding observations it is plain that the following may be regarded as a general method of finding asymptotes. Assume $x = \infty$, or $y = \infty$, and then ascertain whether in each case the value of either of the intercepts x_0, y_0, of the tangent is finite: if one or both of the quantities x_0, y_0, be finite, they will correspond to the existence of an asymptote, the position of which they will define.

Asymptotes of Algebraic Curves.

110. The method of determining the asymptotes of curves, which has been developed in the preceding Article, although always applicable, is not however so convenient in the case of algebraic curves as the one which we shall now propose.

The equation to the tangent at any point (x, y) of a curve may be written in the form

$$x'dy - y'dx = x\,dy - y\,dx,$$

Assume $$x = \frac{\lambda}{r}, \qquad y = \frac{\mu}{r},$$

λ, μ, r, being all regarded as variable. Then the equation to the tangent becomes

$$\begin{aligned} &x'(r\,d\mu - \mu\,dr) - y'(r\,d\lambda - \lambda\,dr) \\ &= \frac{\lambda}{r}(r\,d\mu - \mu\,dr) - \frac{\mu}{r}(r\,d\lambda - \lambda\,dr) \\ &= \lambda\,d\mu - \mu\,d\lambda. \end{aligned}$$

Suppose now that, to render either x or y infinite, we equate r to zero, and let the corresponding values of λ, μ, $\frac{d\lambda}{dr}$, $\frac{d\mu}{dr}$, be denoted by (λ), (μ), $\left(\frac{d\lambda}{dr}\right)$, $\left(\frac{d\mu}{dr}\right)$; then the equation becomes

$$(\lambda)\,y' - (\mu)\,x' = (\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right),$$

which will be the equation to an asymptote, provided that the ratios between (λ), (μ), and $(\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right)$, having been evaluated, it is reduced to the form of the equation to a line passing within a finite distance from the origin.

Remarks on the Equation of the preceding Article.

111. Before proceeding to apply the equation of the preceding Article to particular examples, it is important to shew that the value of

$$(\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right)$$

depends solely upon the product of (λ) or (μ), and a function of the ratio between (λ) and (μ); and that the ratio between (λ) and (μ) may be always determined.

The equation to the curve, when $\frac{\lambda}{r}$, $\frac{\mu}{r}$, are substituted for x, y, respectively, and when negative powers of r have been eradicated, may be written under the form

$$v = \lambda^n F\left(\frac{\mu}{\lambda}\right) + r\lambda^{n-1} f\left(\frac{\mu}{\lambda}\right) + \ldots\ldots = 0,$$

n being the degree of the terms of highest dimensions in λ' and μ, or

$$F\left(\frac{\mu}{\lambda}\right) + \frac{r}{\lambda} f\left(\frac{\mu}{\lambda}\right) + \ldots\ldots = 0.$$

Differentiating with regard to r we see that, r being afterwards equated to zero,

$$0 = F'\left\{\frac{(\mu)}{(\lambda)}\right\} \cdot \frac{(\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right)}{(\lambda)^2} + \frac{1}{(\lambda)} \cdot f\left\{\frac{(\mu)}{(\lambda)}\right\},$$

whence $$(\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right) = -(\lambda)\,\frac{f\left\{\frac{(\mu)}{(\lambda)}\right\}}{F'\left\{\frac{(\mu)}{(\lambda)}\right\}}.$$

Similarly we might shew that

$$(\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right)$$

depends only upon the product of (μ) and a function of the ratio of λ to μ. Now the ratio between (λ) and (μ), putting $r = 0$, is discoverable from the homogeneous equation

$$\phi\,\{(\lambda), (\mu)\} = 0,$$

$\phi(\lambda, \mu)$ representing the terms in v of highest dimensions in λ and μ. Hence the equation for finding asymptotes may, for each asymptote of the curve, be reduced to the equation for a definite straight line.

EXAMPLES.

112. Ex. 1. Take the curve

$$xy^2 - x^3 + 2a^2y = 0.$$

Then $$\lambda\mu^2 - \lambda^3 + 2a^2r^2\mu = 0 \;\ldots\ldots\ldots\ldots (1),$$

and $$\frac{d\lambda}{dr}(\mu - 3\lambda^2) + (2\lambda\mu + 2a^2r^2)\frac{d\mu}{dr} + 4a^2\mu r = 0 \;\ldots\ldots (2).$$

From (1) we see that

$$(\lambda)\,\{(\mu)^2 - (\lambda)^2\} = 0,$$

and from (2), $\left(\frac{d\lambda}{dr}\right)\{(\mu)^2 - 3(\lambda)^2\} + 2(\lambda)(\mu)\left(\frac{d\mu}{dr}\right) = 0.$

Hence we have the three systems

$$\left\{\begin{aligned}&(\lambda) = 0\\ &\left(\frac{d\lambda}{dr}\right) = 0\end{aligned}\right\}, \quad \left\{\begin{aligned}&(\lambda) = (\mu)\\ &\left(\frac{d\lambda}{dr}\right) - \left(\frac{d\mu}{dr}\right) = 0\end{aligned}\right\}, \quad \left\{\begin{aligned}&(\lambda) = -(\mu)\\ &\left(\frac{d\lambda}{dr}\right) + \left(\frac{d\mu}{dr}\right) = 0\end{aligned}\right\}.$$

Hence, from the equation

$$(\lambda)\, y' - (\mu)\, x' = (\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right),$$

we obtain, as equations for the three asymptotes of the curve,

$$x' = 0, \quad y' - x' = 0, \quad y' + x' = 0.$$

Ex. 2. Take the curve

$$xy^2 - x^2y + ay^2 - bx^2 = 0.$$

Then $\qquad \lambda\mu^2 - \lambda^2\mu + ar\mu^2 - br\lambda^2 = 0,$

and $(\mu^2 - 2\lambda\mu - 2br\lambda)\,\frac{d\lambda}{dr} + (2\lambda\mu - \lambda^2 + 2ar\mu)\,\frac{d\mu}{dr} + a\mu^2 - b\lambda^2 = 0.$

Hence we have $\qquad (\lambda)(\mu)\{(\mu) - (\lambda)\} = 0,$

and $\{(\mu)^2 - 2(\lambda)(\mu)\}\left(\frac{d\lambda}{dr}\right) + \{2(\lambda)(\mu) - (\lambda)^2\}\left(\frac{d\mu}{dr}\right)$

$$+ a(\mu)^2 - b(\lambda)^2 = 0.$$

We have therefore the three systems

$$\left\{\begin{aligned}&(\lambda) = 0\\ &\left(\frac{d\lambda}{dr}\right) + a = 0\end{aligned}\right\}, \quad \left\{\begin{aligned}&(\mu) = 0\\ &\left(\frac{d\mu}{dr}\right) + b = 0\end{aligned}\right\}, \quad \left\{\begin{aligned}&(\lambda) = (\mu)\\ -&\left(\frac{d\lambda}{dr}\right) + \left(\frac{d\mu}{dr}\right) + a - b = 0\end{aligned}\right\}.$$

Hence there are three asymptotes corresponding to the equations

$$x' + a = 0, \quad y' + b = 0, \quad y' - x' = b - a.$$

Ex. 3. Take the curve

$$x^2y - x^3 - 3bxy + 2b^2y = 0.$$

Then $\qquad \lambda^2\mu - \lambda^3 - 3br\lambda\mu + 2b^2r^2\mu = 0,$

whence $\qquad (\lambda)^2\{(\mu) - (\lambda)\} = 0 \ldots\ldots\ldots\ldots\ldots (1).$

Also $(2\lambda\mu - 3\lambda^2 - 3br\mu)\dfrac{d\lambda}{dr}$

$$+ (\lambda^2 - 3br\lambda + 2b^2r^2)\frac{d\mu}{dr} - 3b\lambda\mu + 4b^2r\mu = 0 \;....\; (2).$$

From the equation (1) we see that $(\lambda) = 0$, or $(\lambda) = (\mu)$: in the former case the equation (2) becomes nugatory: in order therefore to obtain the required results, we must differentiate this equation with regard to r. Thus, remembering that $r = 0$, $(\lambda) = 0$, we have

$$\left\{2(\mu)\left(\frac{d\lambda}{dr}\right) - 3b(\mu)\right\}\left(\frac{d\lambda}{dr}\right) - 3b(\mu)\left(\frac{d\lambda}{dr}\right) + 4b^2(\mu) = 0,$$

$$\left(\frac{d\lambda}{dr}\right)^2 - 3b\left(\frac{d\lambda}{dr}\right) + 2b^2 = 0,$$

whence $$\left(\frac{d\lambda}{dr}\right) = b, \quad \text{or} \quad \left(\frac{d\lambda}{dr}\right) = 2b.$$

Thus we have two asymptotes represented by the equations

$$x' = b, \quad x' = 2b.$$

Again, supposing that $(\lambda) = (\mu)$, we have, from (2),

$$-\left(\frac{d\lambda}{dr}\right) + \left(\frac{d\mu}{dr}\right) - 3b = 0,$$

so that there is a third asymptote represented by the equation

$$y' - x' = 3b.$$

Algebraical Method of finding Curvilinear and Rectilinear Asymptotes.

113. Let the equation to a curve be reduced if possible to the form $y = f(x)$; and suppose $f(x)$ to be developed in a series of descending powers of x, so that

$$y = a_m x^m + a_{m-1}x^{m-1} + \;....\; + a_1 x + a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \;....$$

Then, when x is indefinitely great, the terms involving negative indices will vanish, and the equation to the curve will ultimately be equivalent to

$$y = a_m x^m + a_{m-1}x^{m-1} + \;....\; + a_1 x + a_0;$$

which is therefore the equation to a curve approaching the proposed curve asymptotically, that is, to a curvilinear asymptote. If m be equal to unity, which is the most important case, the asymptotic equation will designate a rectilinear asymptote. Should there be a rectilinear asymptote parallel to the axis of y, this method will fail to detect it, for the equation $y = a_1 x + a_0$ will not represent straight lines parallel to the axis of y. In order to discover asymptotes parallel to the axis of y, we might obtain a development for x in descending powers of y. It is, however, frequently easy to ascertain by inspection these asymptotes; for, the value of y in the equation $y = f(x)$ being infinite for the abscissa corresponding to such an asymptote, we have only to equate to zero the denominator of $f(x)$: the corresponding values of x being ascertained, the indefinite ordinates belonging to them will be the required asymptotes. This method of finding asymptotes was first given by Stirling, in his *Lineæ Tertii Ordinis Newtonianæ*, p. 48.

Ex. 1. To find the asymptotes of the curve

$$y^2 = \frac{x^3}{x - a}.$$

We have

$$\begin{aligned} y &= \pm x . \left(\frac{x}{x - a}\right)^{\frac{1}{2}} \\ &= \pm x . \left(1 - \frac{a}{x}\right)^{-\frac{1}{2}} \\ &= \pm x \left(1 + \tfrac{1}{2}\frac{a}{x} + \ldots .\right) \\ &= \pm (x + \tfrac{1}{2}a + \ldots .): \end{aligned}$$

hence the equations $y = \pm (x + \frac{1}{2}a)$

determine two rectilinear asymptotes.

Again, $y = \infty$ when x is equal to a: hence

$$x = a$$

is the equation to a third asymptote.

Ex. 2. To find the asymptotes of the curve

$$ay = \frac{x^3}{x - b}.$$

If x be equal to b, y is equal to infinity: thus

$$x = b$$

is the equation to an asymptote parallel to the axis of y.

Again, $$ay = x^2\left(1 - \frac{b}{x}\right)^{-1} = x^2\left(1 + \frac{b}{x} + \frac{b^2}{x^2} + \dots\right),$$

whence it appears that the curve has also a parabolic asymptote, of which the equation is

$$ay = x^2 + bx + b^2.$$

For additional information on the subject of asymptotes, the reader is referred to a paper, by Mr. Gregory, in the *Cambridge Mathematical Journal* for November, 1843, and to a paper in the same Journal for February, 1841.

CHAPTER III.

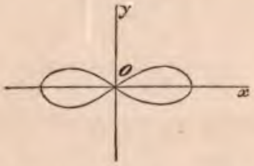
MULTIPLE POINTS, CONJUGATE POINTS, CUSPS, ETC.

Definition.

114. A *multiple point* is a point through which two or more branches of a curve pass. Thus, at the origin of coordinates of the curve of which the equation is

$$y^2 = x^2(1 - x^2),$$

there is a *double* point; that is, a *multiple* point of two branches. The form of this curve is indicated in the following figure:



A *conjugate* or *isolated* point is a point, the coordinates of which satisfy the equation to a curve, while if to either x or y be assigned any value differing ever so little from its value at the point, the corresponding value of y or x respectively will be impossible. Thus, supposing a to be less than c, the curve

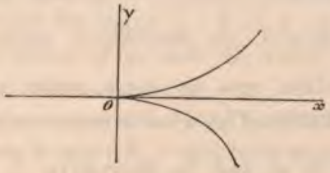
$$(y - b)^2 = (x - a)^2(x^2 - c^2)$$

will have a *conjugate* point, of which the coordinates are (a, b). For, if we put $x = a \pm h$, where h is indefinitely small, $(y - b)^2$ would be equal to a negative quantity, which is impossible so long as y is possible. In like manner, if we put $y = b \pm k$, k being indefinitely small, it appears from the equation to the curve that x cannot have any possible value nearly equal to a.

A *cusp* is a point where two branches of a curve stop abruptly and have a common tangent. Thus, the curve belonging to the equation

$$y^2 = x^3 (a^2 + x^2)$$

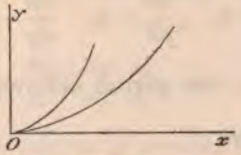
has a cusp at the origin, the common tangent of the two branches coinciding with the axis of x. The following is the form of the curve:



As another example we may take the equation

$$x^4 - ax^2y - axy^2 + \tfrac{1}{4}a^2y^2 = 0,$$

which belongs to a curve of the form



which has a cusp at the origin of coordinates.

There are two species of cusps: the *ceratoid,* so called from its likeness to the horns of cattle, the curvature of the two branches lying in opposite directions, and the *ramphoid,* so called from its likeness to the beak of a bird, the curvature of the two branches lying in the same direction. The former figure affords an instance of a *ceratoid,* the latter of a *ramphoid.*

Analytical Property of Multiple Points in Algebraical Curves.

115. If $u = f(x, y) = 0$ represent the equation to an algebraical curve cleared of radical and negative indices, the values of x and y, at a multiple point, will satisfy the equations

$$\frac{du}{dx} = 0, \qquad \frac{du}{dy} = 0.$$

Let a, b, be the coordinates of a multiple point. It is clear that, since two or more branches pass through the point (a, b), there must be two or more values of y corresponding to the value $a \pm h$ of x, where h is indefinitely small, one value for each branch; and that when $h = 0$, that is, at the point (a, b), all these different values must become equal values; hence it appears that the equation $f(a, y) = 0$ must contain two or more values of y, each equal to b, and that therefore the equation $\frac{d}{dy} f(a, y) = 0$, its derivative, must (by the theory of equations) contain one or more roots, each equal to b. Similarly, the equation $f(x, b) = 0$ must contain two or more roots, each equal to a, and its derivative $\frac{d}{dx} f(x, b) = 0$ must contain one or more roots, each equal to a. Hence the values of x and y, which correspond to a multiple point, must satisfy the equations

$$u = 0, \quad \frac{du}{dx} = 0, \quad \frac{du}{dy} = 0,$$

where $\frac{du}{dx}$ and $\frac{du}{dy}$ are the partial differential coefficients of u with regard to x and y.

COR. Differentiating the equation $u = 0$, we have

$$\frac{du}{dx} + \frac{du}{dy}\,\frac{dy}{dx} = 0;$$

and therefore
$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}};$$

hence at a multiple point $\frac{dy}{dx}$ will present itself under the indeterminate form $\frac{0}{0}$.

Analytical Property of Cusps in Algebraical Curves.

116. The analytical property which we have established in the case of a multiple point holds good also in relation to cusps.

First, let us suppose that the tangent at the cusp is not parallel to either of the coordinate axes. Then it is evident that the very same reasoning is applicable to cusps as to multiple points, in consequence of the common feature which they possess, viz. that the value $a+h$ or $a-h$ of x corresponds to more than one value of y, and the value $b+k$ or $b-k$ of y to more than one value of x.

Secondly, let the tangent at the cusp be parallel to the axis of y. Then, when $y=b+k$ or $b-k$, x will have more than one value, and therefore, when k is equated to zero, and the values of x are thereby made equal to a, the equation $f(x, b)=0$ will have more than one value a of x. It follows, therefore, that at such a cusp $\frac{du}{dx}=0$. Again, differentiating the equation to the curve, we have

$$\frac{du}{dx}+\frac{du}{dy}\frac{dy}{dx}=0:$$

but $\frac{dy}{dx}=\infty$, since the tangent is parallel to the axis of y; and therefore, since $\frac{du}{dx}=0$, it follows that $\frac{du}{dy}$ must also be equal to zero.

Hence, at a cusp as well as a multiple point,

$$u=0, \quad \frac{du}{dx}=0, \quad \frac{du}{dy}=0.$$

Analytical Property of Conjugate Points in Algebraical Curves.

117. The same property may be proved to hold good also for conjugate points.

Let (a, b) be the coordinates of a conjugate point. Then, when $x=a\pm h$, or $y=b\pm k$, h and k being very small quantities, the values of y and x respectively must, by the nature of a conjugate point, be impossible. But, as we know by the theory of equations, impossible roots enter rational equations by pairs, and must therefore, on the alteration of the values of the coefficients, by pairs degenerate into possible ones. Hence, when we put $x = a$, the equation $f(a, y) = 0$ must have at least two equal values b for y; and therefore, by the theory of equations, the equation $\frac{d}{dy} f(a, y) = 0$ must have one of these roots. Similarly, the equations $f(x, b) = 0$ and $\frac{d}{dx} f(x, b) = 0$, must have at least one root a in common. Hence, for the existence of a conjugate point it is necessary that, as in the case of multiple points or cusps,

$$u = 0, \quad \frac{du}{dx} = 0, \quad \frac{du}{dy} = 0.$$

Determination of the Multiplicity and of the Directions of the Tangents at a Multiple Point.

118. Let $$u = f(x_1, y_1) = 0 \; \ldots\ldots\ldots\ldots \; (1)$$

be the equation to a curve free from radicals and negative indices, x_1, y_1, being the coordinates of any point whatever in the curve. Let (x, y) be a multiple point: then at this point we know that

$$\left.\begin{aligned} \frac{du}{dx} &= 0, \\ \frac{du}{dy} &= 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2).$$

Let $x + h$, $y + k$, be another point of the curve near to the multiple point: then, from (1), putting $x + h$, $y + k$, for x_1, y_1, respectively, and expanding $f(x + h, y + k)$ by Taylor's theorem, we have

$$\begin{aligned} 0 = u &+ h\frac{du}{dx} + k\frac{du}{dy} \\ &+ \frac{1}{1.2}\left(h^2\frac{d^2u}{dx^2} + 2hk\frac{d^2u}{dx\,dy} + k^2\frac{d^2u}{dy^2}\right) \\ &+ \frac{1}{1.2.3}\left(h^3\frac{d^3u}{dx^3} + 3h^2k\frac{d^3u}{dx^2\,dy} + 3hk^2\frac{d^3u}{dx\,dy^2} + k^3\frac{d^3u}{dy^3}\right) \\ &+ \&c. \end{aligned}$$

or, by (1), (2),

$$0 = \frac{1}{1.2}\left(h^2\frac{d^2u}{dx^2} + 2hk\,\frac{d^2u}{dx\,dy} + k^2\,\frac{d^2u}{dy^2}\right)$$
$$+ \frac{1}{1.2.3}\left(h^3\frac{d^3u}{dx^3} + 3h^2k\,\frac{d^3u}{dx^2dy} + 3hk^2\,\frac{d^3u}{dx\,dy^2} + k^3\,\frac{d^3u}{dy^3}\right)$$
$$+ \text{\&c.} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Suppose that at the point in question the lowest partial differential coefficients of u, of which at any rate all do not vanish, are of the n^{th} order; then the equation (3) is reduced to

$$0 = \frac{1}{1.2.3\ldots n}\left(h^n\frac{d^nu}{dx^n} + \frac{n}{1}\,h^{n-1}k\,\frac{d^nu}{dx^{n-1}dy} + \frac{n(n-1)}{1.2}\,h^{n-2}k^2\,\frac{d^nu}{dx^{n-2}dy^2}\right.$$
$$\left. + \ldots\ldots + k^n\,\frac{d^nu}{dy^n}\right) + \text{\&c.}$$

Put $h = \lambda x'$, $k = \lambda y'$: then, dividing out by λ^n and multiplying by $1.2.3 \ldots n$, we get

$$0 = x'^n\,\frac{d^nu}{dx^n} + \frac{n}{1}\,x'^{n-1}y'\,\frac{d^nu}{dx^{n-1}dy} + \frac{n(n-1)}{1.2}\,x'^{n-2}y'^2\,\frac{d^nu}{dx^{n-2}dy^2}$$
$$+ \ldots\ldots + y'^n\,\frac{d^nu}{dy^n} + \lambda \text{ (a series of terms).}$$

Now in the limit when $h : k :: dx : dy$, the quantity λ will become less than any assignable quantity, and therefore the equation will ultimately become

$$0 = x'^n\,\frac{d^nu}{dx^n} + \frac{n}{1}\,x'^{n-1}y'\,\frac{d^nu}{dx^{n-1}dy} + \frac{n(n-1)}{1.2}\,x'^{n-2}y'^2\,\frac{d^nu}{dx^{n-2}dy^2}$$
$$+ \ldots\ldots + y'^n\,\frac{d^nu}{dy^n}.$$

This equation, which is homogeneous in x' and y', is equivalent to n linear equations in x' and y', which will represent the tangents to the several branches of the curve, n in number, at the point (x, y), the point (x, y) being considered the origin of coordinates. Thus the degree of plurality of a multiple point is defined by the order of the lowest partial differential coefficients of u which do not vanish.

Ex. 1. To determine the multiplicity of the point $x = 0$, $y = 0$, in the curve
$$u = x^4 + y^4 - axy^2 = 0.$$

At the point in question

$$\frac{du}{dx} = 4x^3 - ay^2 = 0, \quad \frac{du}{dy} = 4y^3 - 2axy = 0,$$

$$\frac{d^2u}{dx^2} = 12x^2 = 0, \quad \frac{d^2u}{dx\,dy} = -2ay = 0, \quad \frac{d^2u}{dy^2} = 12y^2 - 2ax = 0,$$

$$\frac{d^3u}{dx^3} = 24x = 0, \quad \frac{d^3u}{dx^2dy} = 0, \quad \frac{d^3u}{dx\,dy^2} = -2a, \quad \frac{d^3u}{dy^3} = 24y = 0.$$

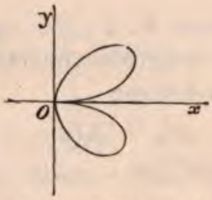
Hence, for the determination of the tangents at the multiple point, which we see is a triple point, we have, substituting for the partial differential coefficients in the equation

$$x'^3 \frac{d^3u}{dx^3} + 3x'^2y' \frac{d^3u}{dx^2dy} + 3x'y'^2 \frac{d^3u}{dx\,dy^2} + y'^3 \frac{d^3u}{dy^3} = 0,$$

$$-6ax'y'^2 = 0,$$

which is equivalent to $x' = 0$, and $y'^2 = 0$:

the former of which equations shews that the axis of y touches one branch of the curve, and the latter, that the tangents to two branches coincide with the axis of x. The form of the curve is exhibited in the diagram:



Ex. 2. To determine the multiple points of the curve

$$(y^2 - 1)^2 = x^2 (2x + 3).$$

In this case $u = (y^2 - 1)^2 - x^2 (2x + 3) = 0$,

and, as conditions for a multiple point,

$$\frac{du}{dx} = -6x^2 - 6x = 0,$$

$$\frac{du}{dy} = 4y (y^2 - 1) = 0.$$

These three equations are satisfied by each of the following systems of values,

$$\begin{Bmatrix} x = -1 \\ y = 0 \end{Bmatrix}, \quad \begin{Bmatrix} x = 0 \\ y = 1 \end{Bmatrix}, \quad \begin{Bmatrix} x = 0 \\ y = -1 \end{Bmatrix}.$$

Proceeding to second differentials, we have

$$\frac{d^2u}{dx^2} = -12x - 6, \quad \frac{d^2u}{dx\,dy} = 0, \quad \frac{d^2u}{dy^2} = 12y^2 - 4.$$

The equation

$$x'^2 \frac{d^2u}{dx^2} + 2x'y' \frac{d^2u}{dxdy} + y'^2 \frac{d^2u}{dy^2} = 0,$$

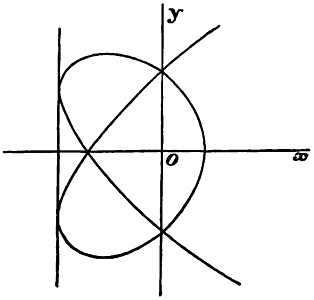
becomes, for the first system of values of x and y,

$$3x'^2 - 2y'^2 = 0,$$

and, for each of the two second systems,

$$-3x'^2 + 4y'^2 = 0.$$

Thus we see that there are three double points. The figure is subjoined.



Multiplicity of a Multiple Point at the Origin.

119. The existence and multiplicity of a multiple point at the origin may be ascertained more simply by inspection. Let the equation to a curve, arranged in groups of terms of different dimensions, be

$$u_m + u_{m-1} + u_{m-2} + \dots\dots + u_n = 0,$$

u_r denoting generally a series of terms of r dimensions, and u_n denoting those of lowest degree in the equation. Then, in the immediate neighbourhood of the origin, we may neglect terms of higher compared with those of lower orders, so that the equation will become

$$u_n = 0,$$

the dimensions of this equation determining the degree of the multiplicity, and the simple factors, into which it may be decomposed, defining, when equated each of them to zero, the directions of the branches. This method of finding the multiplicity of a multiple point, may be readily deduced from the general equation given in the preceding Article.

Ex. 1. Taking the first example of the preceding Article, we have, retaining only the term of the third dimension,

$$xy^2 = 0,$$

which shews that the axis of y touches at the origin one branch, and that of x two branches, of the curve.

Ex. 2. Take the curve

$$x^4 - 2ax^2y - 2x^2y^2 + ay^3 + y^4 = 0.$$

Then, retaining terms of the lowest dimensions, we have

$$-2ax^2y + ay^3 = 0,$$

whence, for the equations to the tangents at the origin,

$$y = 0, \quad y = -2^{\frac{1}{2}}x, \quad \text{or } y = +2^{\frac{1}{2}}x;$$

so that there is a triple point at the origin.

Point of Osculation.

120. A *point of osculation* is a multiple point in which the several branches of the curve have a common tangent. Thus cusps are a species of points of osculation.

Suppose that there are only two branches at the point, then the roots of the equation

$$x'^2\frac{d^2u}{dx^2} + 2x'y'\frac{d^2u}{dx\,dy} + y'^2\frac{d^2u}{dy^2} = 0 \;\ldots\ldots\ldots\; (1),$$

must be equal: hence, as a necessary condition for a point of osculation,

$$\left(\frac{d^2u}{dx\,dy}\right)^2 = \frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2}.$$

If this condition be not satisfied for any point of a curve corresponding to the three equations

$$u = 0,\quad \frac{du}{dx} = 0,\quad \frac{du}{dy} = 0,$$

then we must have either

$$\left(\frac{d^2u}{dx\,dy}\right)^2 > \frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2},$$

in which case there will be a double point with two distinct tangents; or

$$\left(\frac{d^2u}{dx\,dy}\right)^2 < \frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2},$$

when the equation (1) will give impossible relations between x' and y', and (x, y) will be the coordinates of a conjugate point.

Remark on the Theory of Multiple Points.

121. If there be a multiple point in a curve, its position and its multiplicity may be ascertained by investigating the pairs of values of x and y which satisfy the three equations

$$u = 0,\quad \frac{du}{dx} = 0,\quad \frac{du}{dy} = 0,$$

and by determining the order of the lowest partial differential coefficients of u which do not all vanish at the point. The directions of the tangents will be ascertained by the formula of Art. (118). If the relations between x' and y', expressed by this formula, be impossible, the existence of a conjugate point is at once indicated. We cannot however be sure that the point is not a conjugate instead of a multiple point or cusp, even when all the relations between x' and y' are possible. Additional considerations are necessary in order to ascertain this to a certainty: an examination of the general nature of the curve in the neighbourhood of the point, by an algebraical discussion of its equation, is sufficient for the purpose.

Ex. Take the curve

$$(y - cx^3)^2 = (x - a)^6 . (x - b)^5,$$

a being supposed to be less than b. Then, putting

$$u = (y - cx^3)^2 - (x - a)^6 . (x - b)^5 = 0,$$

$$\frac{du}{dx} = - 6cx^2 (y - cx^3) - 6 (x - a)^5 (x - b)^5 - 5 (x - a)^6 (x - b)^4 = 0,$$

$$\frac{du}{dy} = 2 (y - cx^3) = 0,$$

we see that $x = a, \quad y = a^3 c.$

Then also $\dfrac{d^2u}{dx^2} = 18a^4c^2, \quad \dfrac{d^2u}{dx\,dy} = - 6a^2c, \quad \dfrac{d^2u}{dy^2} = 2,$

and therefore, from the formula

$$x'^2 \frac{d^2u}{dx^2} + 2x'y' \frac{d^2u}{dx\,dy} + y'^2 \frac{d^2u}{dy^2} = 0,$$

there is

$$y'^2 - 6a^2cx'y' + 9a^4c^2x'^2 = 0,$$

$$(y' - 3a^2cx')^2 = 0,$$

which might seem to indicate a point of osculation, when the equation to the common tangent of the two branches would be

$$y' = 3a^2cx'.$$

It is easily seen, however, that the point is really a conjugate point. For

$$y = cx^3 \pm (x - a)^3 (x - b)^{\frac{5}{2}},$$

which shews that y is impossible when x differs very slightly from a.

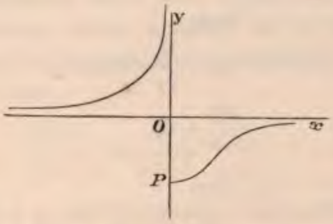
For further information on the subject of this Article, the reader is referred to a paper on the General Theory of Multiple Points in the *Cambridge Mathematical Journal* for November, 1840.

Points d'Arrêt or Points de Rupture.

122. An algebraic curve never stops abruptly in its course, that is, it never possesses singular points of the kind called by French writers *points d' arrêt* or *de rupture*. Such points are however of frequent occurrence in transcendental curves. For instance, in the curve belonging to the equation

$$y + 1 = e^{-\frac{1}{x}},$$

the form of which is



there is a *point d' arrêt* at P.

The impossibility of the existence of such points in curves represented by algebraical equations depends upon the fact that impossible roots enter algebraical equations, involving one unknown letter, by pairs. Suppose in fact that, when x is equal to $a - h$, y has an impossible value for each small value of h however small, and that y has a possible value when x is equal to $a + h$. Then, when x passes from $a - h$ to $a + h$, one value of y, and therefore, by the nature of algebraical equations, two values of y, the two values being of the forms $\alpha \pm \sqrt{(-\beta)}$, must change into possible ones, which will evidently, in consequence of the correspondency of their values, be equal to each other when h is indefinitely diminished. The existence of two equal values of y, corresponding to the value a of x, shews that there is no abrupt termination of the curve at the point of which the abscissa is a.

Points Saillants.

123. A *point saillant* is a point of a curve where two branches of the curve stop abruptly and have tangents inclined to each

other at a finite angle. Such points are frequently to be met with in transcendental curves, but can have no existence in curves corresponding to algebraical equations.

Ex. 1. Take the curve of which the equation is

$$y = \frac{x}{1 + e^{\frac{1}{x}}}.$$

Then
$$\frac{dy}{dx} = \frac{1}{1 + e^{\frac{1}{x}}} + \frac{1}{x \,.\, e^{-\frac{1}{x}} \,.\, (1 + e^{\frac{1}{x}})^2}.$$

Suppose that $x = +\,0$: then

$$y = \frac{0}{\infty} = 0,$$

and
$$\frac{dy}{dx} = 0.$$

Suppose that $x = -\,0$: then

$$y = \frac{0}{1} = 0,$$

and
$$\frac{dy}{dx} = 1.$$

The origin of coordinates is therefore a *point saillant:* the branch corresponding to positive values of the abscissæ touches the axis of x, while the tangent to the other branch at the same point is inclined at an angle of 45° to this axis.

Ex. 2. Take the curve

$$y = x \tan^{-1} \frac{1}{x}.$$

There will be a *point saillant* at the origin of coordinates: the directions of the two branches at this point being defined by the values $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$ of $\frac{dy}{dx}$.

We have observed that an algebraical curve does not admit of *points saillants.* This may readily be shewn. Suppose, in fact, that between the equation to the curve

$$u = 0,$$

and its derivative
$$\frac{du}{dx} + \frac{du}{dy} \,.\, y' = 0,$$

we eliminate y: we may then obtain an algebraical equation between x and y', free from radicals and fractional forms. If then we conceive a curve to be constructed, of which the abscissa shall be always equal to x and the ordinate to y', this curve can have no *point d'arrêt*, which would necessarily be the case if the primitive curve had any *point saillant.*

Branches Pointillées.

124. We occasionally meet with equations, the geometrical loci of which consist, either entirely or in part, of a series of conjugate points, forming *branches pointillées.*

Ex. Take the curve

$$\left(\frac{y}{a}\right)^2 = \log\left\{\sin\left(\frac{x}{a}\right)\right\}.$$

It is evident that y will be always impossible when $\sin\left(\frac{x}{a}\right)$ has any negative value or any positive value less than unity; and therefore, unity being the greatest value of the sine of an angle, we must have

$$\frac{x}{a} = (4\lambda + 1)\frac{\pi}{2}, \quad y = 0,$$

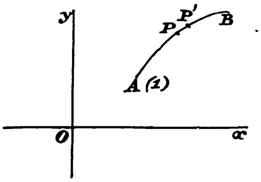
λ being an integer. Thus we see that the geometrical locus of the equation consists of a series of conjugate points arranged along the axis of x at a common interval equal to $2\pi a$, the axis of abscissæ being thus a *branche pointillée.*

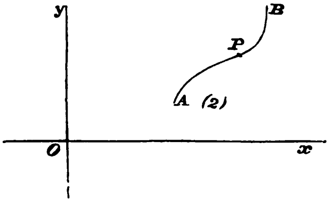
CHAPTER IV.

CONCAVITY AND CONVEXITY OF CURVES AND POINTS OF INFLECTION.

Conditions for Concavity and Convexity.

125. THE object of this chapter is to investigate the condition that a curve, of which the equation is given, may at any assigned point turn its concavity or convexity towards either of the coordinate axes, and to determine those peculiar points of the curve, called *points of inflection* or *of contrary flexure*, at which concavity is succeeded by convexity, or conversely.





In fig. (1) the curve AB is concave at P towards the axis of x, and convex towards that of y; in fig. (2) there is a point of inflection at P, the curve being concave towards the axis of x at each point in the arc AP, and convex at each point of the arc PB.

Let Ψ, Ψ', be the inclinations of the tangents of the curve at P, P', in fig. (1) to the axis of x, P' being a point near to P: then, as we pass through P from A towards B, it is evident that $\tan\Psi$ will keep continuously decreasing: hence, x, x', being the coordinates of P, P',

$$\frac{\tan\Psi' - \tan\Psi}{x' - x} = \text{a negative quantity}:$$

and this will be true however near P' may be to P; and therefore, proceeding to the limit, we see that, as the condition of concavity, $\dfrac{d \tan \psi}{dx}$ = a negative quantity.

If the curve were convex at P towards the axis of x, it is evident from like reasoning that

$$\frac{d \tan \psi}{dx} = \text{a positive quantity.}$$

But $\tan \psi = \dfrac{dy}{dx}$: hence, that the curve may be concave towards the axis of x at a point x, y, it is necessary that $\dfrac{d^2 y}{dx^2}$ be negative, and, that it may be convex, that $\dfrac{d^2 y}{dx^2}$ be positive.

It is easily seen that the conditions which we have shewn to be necessary are also sufficient conditions for concavity and convexity. For it is evident that the curve will be concave or convex towards the axis of x at a point (x, y) accordingly as

$$\frac{\tan \psi' - \tan \psi}{x' - x}$$

is negative or positive for all indefinitely small values of ψ' nearly equal to ψ, that is, proceeding to the limit, as $\dfrac{d^2 y}{dx^2}$ is negative or positive.

If y be negative, or the point P be on the opposite side of the axis of x, then, as may be seen simply by changing the sign of y throughout, we must have, for concavity,

$$\frac{d^2 y}{dx^2} = \text{a positive quantity;}$$

and, for convexity,

$$\frac{d^2 y}{dx^2} = \text{a negative quantity.}$$

Without regarding the sign of y, we may state generally that the sufficient and necessary condition for concavity is that

$$y \frac{d^2 y}{dx^2} = \text{a negative quantity,}$$

and, for convexity, that

$$y\frac{d^2y}{dx^2} = \text{a positive quantity.}$$

Ex. To find where the Witch of Agnesi, of which the equation is

$$y^2 = 4a^2\frac{2a-x}{x},$$

is concave or convex towards the axis of x.

Differentiating, we have

$$y\frac{dy}{dx} = -\frac{4a^3}{x^2},$$

$$y\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} = \frac{8a^3}{x^3},$$

or, multiplying by y^2 and replacing $y^2\frac{dy^2}{dx^2}$ by its value in terms of x,

$$y^3\frac{d^2y}{dx^2} + \frac{16a^6}{x^4} = \frac{8a^3}{x^3}\cdot 4a^2\frac{2a-x}{x},$$

whence $$y^3\frac{d^2y}{dx^2} = \frac{16a^5}{x^4}(3a-2x).$$

This result shews that $y\frac{d^2y}{dx^2}$ is negative when x is greater, and positive when x is less than $\frac{3a}{2}$. Thus from $x = 0$ to $x = \frac{3a}{2}$ the curve is convex towards the axis of x, and concave from $x = \frac{3a}{2}$ to $x = 2a$. There must accordingly be two points of inflection corresponding to the value $2a$ of x.

Condition for a Point of Inflection.

126. Since a point of inflection separates two portions of a curve, one of which is concave and the other convex towards the axis of x, it follows that, as x keeps continuously increasing, $\tan\psi$ or $\frac{dy}{dx}$ must either keep increasing as we approach the point and decreasing as we recede from it, or conversely.

Hence, that there may be inflection at any proposed point, it is sufficient and necessary that $\frac{dy}{dx}$ have a maximum value, or that $\frac{d^2y}{dx^2}$ experience a change of sign as we pass along the curve from one side of the point to the other.

In order then to determine points of inflection, we have only to ascertain those values of x and y which render $\frac{dy}{dx}$ either a maximum or a minimum.

Ex. 1. To find the point of inflection in the curve

$$x^4 - a^2x^2 + a^3y = 0.$$

Differentiating twice, we get

$$4x^3 - 2a^2x + a^3\frac{dy}{dx} = 0,$$

$$12x^2 - 2a^2 + a^3\frac{d^2y}{dx^2} = 0.$$

Putting $\frac{d^2y}{dx^2} = 0$, in order to get the values of x which correspond to the maximum or minimum values of $\frac{dy}{dx}$, we have

$$x = \pm\frac{a}{\sqrt{6}},$$

and therefore $$y = \frac{5a}{36}.$$

Differentiating a third time, we see that

$$24x + a^3\frac{d^3y}{dx^3} = 0;$$

which shews that $\frac{d^3y}{dx^3}$ has a finite value when $\frac{d^2y}{dx^2} = 0$. Hence the values $\pm\frac{a}{\sqrt{6}}$ of x make $\frac{dy}{dx}$ a maximum or minimum, and therefore correspond to two points of inflection.

Ex. 2. To find the points of inflection of the curve

$$x^4 = a^2y^2 + x^2y^2.$$

We have
$$y = \pm \frac{x^2}{(x^2 + a^2)^{\frac{1}{2}}},$$
$$\frac{dy}{dx} = \pm \frac{x^3 + 2a^2x}{(x^2 + a^2)^{\frac{3}{2}}},$$
$$\frac{d^2y}{dx^2} = \pm \frac{a^2(2a^2 - x^2)}{(x^2 + a^2)^{\frac{5}{2}}}.$$

As far as signs are concerned, we may take v instead of $\frac{d^2y}{dx^2}$, where
$$v = \pm (2a^2 - x^2).$$

When $v = 0$, we have $x = \pm a\sqrt{2}$, and
$$\frac{dv}{dx} = \mp 2x = \text{a finite quantity}.$$

Hence the values $\pm a\sqrt{2}$ of x correspond to maximum or minimum values of $\frac{dy}{dx}$, and therefore to points of inflection.

Ex. 3. To ascertain whether the Lemniscata of which the equation is
$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$
has a point of inflection at the origin of coordinates.

Differentiating twice, we have
$$2(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) = a^2\left(x - y\frac{dy}{dx}\right),$$
$$4\left(x + y\frac{dy}{dx}\right)^2 + 2(x^2 + y^2)\left(1 + y\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2}\right)$$
$$= a^2\left(1 - y\frac{d^2y}{dx^2} - \frac{dy^2}{dx^2}\right).$$

From the last equation we see that, when $x = 0$ and $y = 0$,
$$\frac{dy}{dx} = \pm 1,$$
the value of $\frac{d^2y}{dx^2}$ remaining indeterminate. In order to find $\frac{d^2y}{dx^2}$, we must differentiate again.

Thus

$$12\left(x + y\frac{dy}{dx}\right)\left(1 + y\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2}\right) + 2\left(x^2 + y^2\right)\left(1 + 3\frac{dy}{dx}\frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3}\right) = -3a^2\frac{dy}{dx}\frac{d^2y}{dx^2} - a^2y\frac{d^3y}{dx^3}.$$

When $x = 0$ and $y = 0$, we see that $\frac{d^2y}{dx^2} = 0$. In order to determine the corresponding value of $\frac{d^3y}{dx^3}$, we must proceed to another differentiation: we then have, omitting all those terms which vanish when x and y are both zero,

$$16\left(1 + \frac{dy^2}{dx^2}\right)^2 = -4a^2\frac{dy}{dx}\frac{d^3y}{dx^3},$$

whence

$$\frac{d^3y}{dx^3} = \mp\frac{16}{a^2}.$$

We have shewn therefore that $\frac{d^2y}{dx^2}$ has a zero value, and $\frac{d^3y}{dx^3}$ a finite one, at the origin of coordinates, whether we take $\frac{dy}{dx}$ equal to $+1$ or to -1. Hence both branches of the curve have inflection at the origin.

Symmetrical Investigation of Points of Inflection.

127. A point of inflection being an absolute peculiarity in a curve, and not, like a point of concavity or convexity, having especial reference to either axis in particular, it is desirable to develop also a method of determining such a point which shall be symmetrically related to both axes.

Let* the equation to an algebraical curve, cleared of radicals and negative indices, be represented by

$$F = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

where F is a rational function of x and y.

Let ds denote an element of the arc of the curve at the point x, y, and let s be taken as the independent variable.

Let
$$U = \frac{dF}{dx}, \quad V = \frac{dF}{dy},$$

$$u = \frac{d^2F}{dx^2}, \quad w = \frac{d^2F}{dx\,dy}, \quad v = \frac{d^2F}{dy^2},$$

$$l = \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad l' = \frac{dl}{ds}, \quad m' = \frac{dm}{ds}.$$

Then, differentiating (1), we have

$$lU + mV = 0 \ldots\ldots\ldots\ldots\ldots \text{(2)};$$

differentiating (2),

$$l^2u + 2lmw + m^2v + l'U + m'V = 0 \ldots\ \ldots. \text{(3)}.$$

Again, it is clear that $l^2 + m^2 = 1$, and therefore

$$ll' + mm' = 0 \ldots\ldots\ldots\ldots\ldots \text{(4)}.$$

From (2) and (4), we get

$$l'V = m'U;$$

hence, multiplying (3) by U, we obtain

$$U(l^2u + 2lmw + m^2v) + l'(U^2 + V^2) = 0,$$

or, by virtue of (2),

$$\frac{l^2U}{V^2}(V^2u - 2UVw + U^2v) + l'(U^2 + V^2) = 0 \ldots. \text{(5)}.$$

In a similar way we may shew that

$$\frac{m^2V}{U^2}(V^2u - 2UVw + U^2v) + m'(U^2 + V^2) = 0 \ldots. \text{(6)}.$$

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* This method of determining multiple points was published in the *Cambridge Mathematical Journal* for November, 1843.

Again, by the relation $l^2 + m^2 = 1$ and the equation (2), we get

$$\frac{l^2}{V^2} = \frac{1}{U^2 + V^2} = \frac{m^2}{U^2}:$$

hence (5) and (6) give us

$$U(V^2u - 2UVw + U^2v) + l'(U^2 + V^2)^2 = 0 \ldots\ldots (7),$$

$$V(V^2u - 2UVw + U^2v) + m'(U^2 + V^2)^2 = 0 \ldots\ldots (8).$$

Now at a point of inflection it is evident that l and m must be the one a maximum and the other a minimum: hence, as we pass in the neighbourhood of the point along the curve from one side of the point to the other, we know by the theory of maxima and minima that l' and m' must each of them change sign. It is evident then, from (7) and (8), that

$$U(V^2u - 2UVw + U^2v),$$

and

$$V(V^2u - 2UVw + U^2v),$$

must both change sign.

Suppose first that neither U nor V changes sign as we pass through the point; then the sufficient and necessary condition for a change of sign in the value of l' and m', is that

$$V^2u - 2UVw + U^2v \ldots\ldots\ldots\ldots\ldots (9)$$

change sign as we pass through the point: this condition evidently involves the fact that at the point itself

$$V^2u - 2UVw + U^2v = 0 \ldots\ldots\ldots\ldots (10).$$

Next suppose that U changes sign; then it is evident that the expression (9) must not change sign, for otherwise l', as will be evident from the formulæ (7), could not change sign. But from (8) we see that for a change of sign in the value of m', one and one only of the quantities V and (9) must change sign; hence V only must change sign. Thus we see that if either of the quantities U and V change sign, both must do so, and that (9) must not change sign. If U and V both change sign it is clear that at the point itself

$$U = 0, \quad V = 0,$$

which are the conditions for multiplicity of branches at the point.

The general rule, therefore, for finding points of inflection may be thus enunciated. First ascertain the values of x and y which will satisfy simultaneously the equations

$$F = 0,\ V^2u - 2UVw + U^2v = 0,$$

and reject all the pairs of values thus obtained which do not, as we pass through the corresponding points, correspond to a change of sign in the expression

$$V^2u - 2UVw + U^2v,$$

or which do correspond to a change of sign in either U or V: the pairs of values of x and y, which are retained, correspond to points of inflection.

Secondly, ascertain those pairs of values which satisfy simultaneously

$$F = 0, \quad U = 0, \quad V = 0,$$

and reject all of these pairs which do not correspond to a change of sign in both U and V as we pass through the corresponding points along one or other of the branches, or which do correspond to a change of sign in the expression

$$V^2u - 2UVw + U^2v.$$

In the preceding investigation we have supposed F to be a rational function of x and y. Should this not be the case it will be evident, from what has been said, that in addition to the values of x and y, which may be obtained by the rule which we have enunciated, we must likewise take those which will render in the first case,

$$F = 0, \quad V^2u - 2UVw + U^2v = \infty;$$

and, in the second case,

$$F = 0, \quad U = \infty, \quad V = \infty;$$

the conditions depending on change of sign being the same as before.

Ex. 1. Let the curve be

$$F = ax^3 + by^3 - c^4 = 0.$$

Then
$$U = 3ax^2, \quad V = 3by^2,$$
$$u = 6ax, \quad w = 0, \quad v = 6by.$$

Hence, from the formula (10) there is, if we cast out constant factors,

$$xy\,(ax^3 + by^3) = 0;$$

or, by the equation to the curve,

$$xy = 0.$$

Thus $x = 0$, or $y = 0$, and in both cases neither U nor V changes sign, while the formula (9) does change sign. Hence we have two points of inflection,

$$x = 0, \quad y = \frac{c^{\frac{4}{3}}}{b^{\frac{1}{3}}},$$

and $$x = \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}, \quad y = 0.$$

Ex. 2. Take the curve

$$F = (x^2 + y^2)^2 - a^2x^2 + b^2y^2 = 0,$$

and suppose that we wish to find whether there be a point of inflection at the origin. Then

$$U = 2x\,(2x^2 + 2y^2 - a^2),$$
$$V = 2y\,(2x^2 + 2y^2 + b^2),$$
$$u = 12x^2 + 4y^2 - 2a^2,$$
$$v = 4x^2 + 12y^2 + 2b^2,$$
$$w = 8xy.$$

From these results it is evident that U and V both change sign if we change x and y each of them from $\pm\, 0$ to $\mp\, 0$. Moreover it is clear that neither V^2u, U^2v, nor $2\,UVw$, experience any change of sign when we put $\pm\, x$, $\pm\, y$, for $\mp\, x$, $\mp\, y$, respectively. Hence the expression (9) does not change sign. If we had kept y positive or negative throughout, while we changed x from $\pm\, 0$ to $\mp\, 0$, the expression (9) would have changed sign, and flexure would not have taken place. Hence we see that the branch which passes through the origin from below to above the axis of x, or that which passes from above to below, will have an inflection at the origin.

Ex. 3. Let the curve be

$$F = \left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} - 1 = 0.$$

Then
$$U = \frac{1}{3a}\left(\frac{x}{a}\right)^{-\frac{2}{3}}, \quad V = \frac{1}{3b}\left(\frac{y}{b}\right)^{-\frac{2}{3}},$$

$$u = -\frac{2}{9a^2}\left(\frac{x}{a}\right)^{-\frac{5}{3}}, \quad w = 0, \quad v = -\frac{2}{9b^2}\left(\frac{y}{b}\right)^{-\frac{5}{3}}.$$

Hence the expression

$$V^2u - 2UVw + U^2v$$

will vary as

$$\left(\frac{x}{a}\right)^{-\frac{5}{3}} . \left(\frac{y}{b}\right)^{-\frac{4}{3}} + \left(\frac{y}{b}\right)^{-\frac{5}{3}} . \left(\frac{x}{a}\right)^{-\frac{4}{3}}$$

$$= \left(\frac{x}{a}\right)^{-\frac{5}{3}} . \left(\frac{y}{b}\right)^{-\frac{5}{3}} . \left\{\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}}\right\}$$

$$= \left(\frac{x}{a}\right)^{-\frac{5}{3}} . \left(\frac{y}{b}\right)^{-\frac{5}{3}}.$$

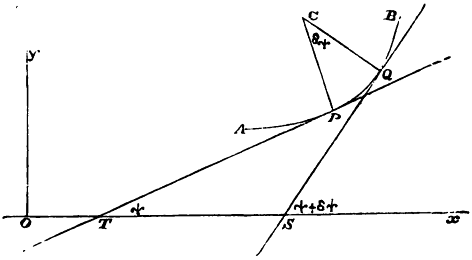
Putting this expression $= \infty$, we get $x = 0$, or $y = 0$; and therefore, by the equation to the curve, $y = b$, $x = a$, respectively. It is evident, then, that as x passes through 0, U and V do not change sign while the expression (9) does; and similarly for y: hence there are two points of inflection, viz. $x = 0$, $y = b$, and $x = a$, $y = 0$.

CHAPTER V.

ON THE INDEX OF CURVATURE, THE RADIUS OF CURVATURE, AND THE CENTRE OF CURVATURE, OF A PLANE CURVE.

Index of Curvature.

128. LET ψ, $\psi + \delta\psi$, be the inclinations of the tangents PT, QS, at points P, Q, of a curve AB, very near to each other, to the axis of x. Then it is evident that the greater be the angle $\delta\psi$ between the two tangents, for a given small arc PQ, or the smaller be the arc PQ for an assigned value of the small angle $\delta\psi$, the greater will be the curvature of the curve in the vicinity of P. Hence, proceeding to the limit, when the arc PQ or δs becomes less than any assignable magnitude, we see that $\frac{d\psi}{ds}$ is a measure of the curvature at P. The angle $\delta\psi$ is called the *angle of contingence*, and $\frac{d\psi}{ds}$ *the index of curvature at P.*



Radius and Centre of Curvature.

129. From P and Q draw two normals PC, QC, meeting in C. These two normals will evidently include an angle $\delta\psi$.

Let $PC = \rho$, $QC = \rho'$, the chord of $PQ = c$, and let $\frac{1}{2}\pi - a$ denote the angle between the chord of PQ and the normal QC. Then it is plain that

$$c \sin(\tfrac{1}{2}\pi - a) = \rho \sin \delta\psi,$$

$$\frac{\sin \delta\psi}{c} = \frac{\sin(\frac{1}{2}\pi - a)}{\rho}:$$

but, proceeding to the limit, when $\delta\psi$, c, and a, become less than any assignable quantities, $\sin \delta\psi$ and c vanish in a ratio of equality with $\delta\Psi$ and δs respectively: hence

$$\frac{d\psi}{ds} = \frac{1}{\rho} \dots\dots\dots\dots\dots\dots (1).$$

Now, ρ and ρ' being ultimately in a ratio of equality, it follows that a circle described with C as a centre, and touching the line PT in P, will ultimately touch the line QS in Q. The angle of contingence will accordingly be the same ultimately for the circle as for the curve: also the arc PQ in the circle will be ultimately in a ratio of equality with the arc PQ in the curve, since each of these arcs, by the 7th Lemma of Newton's Principia, vanishes in a ratio of equality with their common chord. Hence a circle so described has the same curvature as the curve at the point P. This circle is called the *osculating circle*, or the *circle of curvature at the point P*, ρ the *radius*, and C the *centre of curvature.*

The equation (1) shews that the index of curvature at any point of a curve is equal to the reciprocal of the radius of the osculating circle.

Expression for ρ when x *is the Independent Variable.*

130. Differentiating the equation (Art. 100)

$$\tan \psi = \frac{dy}{dx},$$

considering x the independent variable, we have

$$\sec^2 \psi \frac{d\psi}{ds} = \frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}}:$$

but, by Art. (100),

$$\cos\psi = \frac{dx}{ds}, \quad \frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2};$$

hence
$$\left(\frac{d\psi}{ds}\right)^2 = \frac{\left(\dfrac{d^2y}{dx^2}\right)^2}{\dfrac{ds^6}{dx^6}} = \frac{\left(\dfrac{d^2y}{dx^2}\right)^2}{\left(1 + \dfrac{dy^2}{dx^2}\right)^3},$$

and therefore
$$\rho^2 = \frac{\left(1 + \dfrac{dy^2}{dx^2}\right)^3}{\left(\dfrac{d^2y}{dx^2}\right)^2}.$$

From this formula we perceive that the index of curvature will become zero, and the radius of curvature infinite, whenever $\frac{d^2y}{dx^2}$ is equal to zero: the osculating circle will then degenerate into a straight line and coalesce with the tangent. Such will be the case, for instance, at points of inflection, where $\frac{d^2y}{dx^2} = 0$ and $\frac{dy}{dx}$ is not infinite. If, at any point of the curve, $\frac{d^2y}{dx^2}$ becomes infinite, while $\frac{dy}{dx}$ is either zero or of finite magnitude, the index of curvature will become infinite, and the radius of curvature will vanish. If the quantities $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ become simultaneously infinite, the expression for ρ^2 will assume the form $\frac{\infty}{\infty}$; its real value must then be estimated by the rules for the evaluation of indeterminate functions. If one of the functions $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, becomes discontinuous, and experiences an abrupt change of value, such will also be the nature of the index of curvature: such a peculiarity will present itself, for example, at a *point saillant*.

Ex. 1. To find the radius of curvature at any point of the curve
$$y^2 = 4mx.$$

We obtain, by differentiating the equation

$$y = 2m^{\frac{1}{2}}x^{\frac{1}{2}},$$

$$\frac{dy}{dx} = m^{\frac{1}{2}}x^{-\frac{1}{2}}, \qquad \frac{d^2y}{dx^2} = -\tfrac{1}{2}m^{\frac{1}{2}}x^{-\frac{3}{2}}.$$

Hence
$$\rho^2 = \frac{(1 + mx^{-1})^3}{\frac{1}{4}mx^{-3}} = \frac{4(x+m)^3}{m}.$$

If $x = 0$, then $\rho^2 = 4m^2$, $\rho = 2m$; which shews that the radius of curvature at the vertex of a common parabola is equal to half its *latus rectum*.

Expressions for ρ *when* s *is the Independent Variable.*

131. Since, by Art. (100),

$$\cos\psi = \frac{dx}{ds} \quad \text{and} \quad \sin\psi = \frac{dy}{ds},$$

we see that, s being the independent variable,

$$-\sin\psi\,\frac{d\psi}{ds} = \frac{d^2x}{ds^2},$$

and
$$\cos\psi\,\frac{d\psi}{ds} = \frac{d^2y}{ds^2}:$$

hence, squaring and adding these two last equations,

$$\frac{1}{\rho^2} = \frac{d\psi^2}{ds^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

Also
$$\frac{d\psi}{ds} = -\frac{\frac{d^2x}{ds^2}}{\sin\psi} = -\frac{\frac{d^2x}{ds^2}}{\frac{dy}{ds}},$$

whence
$$\rho = -\frac{\frac{dy}{ds}}{\frac{d^2x}{ds^2}}:$$

similarly we may shew that

$$\rho = \frac{\dfrac{dx}{ds}}{\dfrac{d^2y}{ds^2}}.$$

Expression for ρ in terms of dx, dy, d²x, d²y.

132. Since $\tan\psi = \dfrac{dy}{dx}$, we have

$$\psi = \tan^{-1}\left(\frac{dy}{dx}\right),$$

and therefore
$$d\psi = \frac{dx\, d^2y - dy\, d^2x}{dx^2 + dy^2}:$$

$$\frac{1}{\rho^2} = \frac{d\psi^2}{ds^2} = \frac{(dx\, d^2y - dy\, d^2x)^2}{(dx^2 + dy^2)^3}.$$

Expression for ρ in terms of Partial Differential Coefficients.

133. Let $u = 0$ be the equation to a curve. Differentiating this equation twice we get

$$\frac{du}{dx}\, dx + \frac{du}{dy}\, dy = 0 \ldots\ldots\ldots\ldots (1),$$

$$-\left(\frac{du}{dx}\, d^2x + \frac{du}{dy}\, d^2y\right) = \frac{d^2u}{dx^2}\, dx^2 + 2\,\frac{d^2u}{dx\, dy}\, dx\, dy + \frac{d^2u}{dy^2}\, dy^2 \ldots (2).$$

Multiplying (2) by dy, dx, successively, and in each case availing ourselves of the relation (1), we get

$$\frac{du}{dx}\,(dx\, d^2y - dy\, d^2x) = dy\left(\frac{d^2u}{dx^2}\, dx^2 + 2\,\frac{d^2u}{dx\, dy}\, dx\, dy + \frac{d^2u}{dy^2}\, dy^2\right),$$

$$\frac{du}{dy}\,(dy\, d^2x - dx\, d^2y) = dx\left(\frac{d^2u}{dx^2}\, dx^2 + 2\,\frac{d^2u}{dx\, dy}\, dx\, dy + \frac{d^2u}{dy^2}\, dy^2\right):$$

adding together the squares of these two equations, we obtain

$$\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right)(dx\, d^2y - dy\, d^2x)^2$$
$$= (dx^2 + dy^2)\left(\frac{d^2u}{dx^2}\, dx^2 + 2\,\frac{d^2u}{dx\, dy}\, dx\, dy + \frac{d^2u}{dy^2}\, dy^2\right)^2:$$

whence, by the formula of Art. (132),

$$\frac{1}{\rho^2} = \frac{\left(\frac{d^2u}{dx^2}\,dx^2 + 2\,\frac{d^2u}{dx\,dy}\,dx\,dy + \frac{d^2u}{dy^2}\,dy^2\right)^2}{(dx^2 + dy^2)^2\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right)},$$

and therefore, replacing dx, dy, by the quantities $\frac{du}{dy}$, $-\frac{du}{dx}$, to which, by virtue of (1), they are proportional, we get

$$\frac{1}{\rho^2} = \frac{\left(\frac{du^2}{dy^2}\,\frac{d^2u}{dx^2} - 2\,\frac{du}{dx}\,\frac{du}{dy}\,\frac{d^2u}{dx\,dy} + \frac{du^2}{dx^2}\,\frac{d^2u}{dy^2}\right)^2}{\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right)^3}.$$

Cor. If the function u consist of two parts, of which one contains x alone, and the other y alone,

$$\frac{d^2u}{dx\,dy} = 0,$$

and therefore
$$\frac{1}{\rho^2} = \frac{\left(\frac{du^2}{dy^2}\,\frac{d^2u}{dx^2} + \frac{du^2}{dx^2}\,\frac{d^2u}{dy^2}\right)^2}{\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2}\right)^3}.$$

Ex. To find the radius of curvature at any point of the curve

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Here
$$\frac{du}{dx} = \frac{2x}{a^2}, \quad \frac{du}{dy} = \frac{2y}{b^2},$$

$$\frac{d^2u}{dx^2} = \frac{2}{a^2}, \quad \frac{d^2u}{dx\,dy} = 0, \quad \frac{d^2u}{dy^2} = \frac{2}{b^2}:$$

hence
$$\frac{1}{\rho^2} = \frac{\left(\frac{y^2}{a^2b^4} + \frac{x^2}{b^2a^4}\right)^2}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^3} = \frac{1}{a^4b^4\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^3},$$

$$\rho^2 = a^4b^4\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^3.$$

Another method of finding the Radius of Curvature.

134. Let i denote the distance of Q, in the figure, from the line CP, and δ its distance from the tangent at P. Then, since ultimately the circle of curvature touches PT, QS, at P, Q, we have, by the nature of a circle,

$$\text{limit of } i^2 = \text{limit of } (2\rho - \delta)\,.\,\delta,$$

or, since δ vanishes in the limit when compared with ρ,

$$\rho = \text{limit of } \frac{i^2}{2\delta}.$$

Ex. 1. To find the radius of curvature at the vertex of a parabola

$$y^2 = 4mx.$$

Here
$$\rho = \text{limit of } \frac{y^2}{2x} = 2m.$$

Ex. 2. To find the radii of curvature at the extremities of the axes of an ellipse.

If a and b be its semi-axes, and if the major axis and the tangent at one of its extremities be taken as axes of coordinates, its equation will be

$$y^2 = \frac{b^2}{a^2}(2ax - x^2).$$

Hence
$$\rho = \text{limit of } \frac{y^2}{2x} = \frac{b^2}{a^2}(a - \tfrac{1}{2}x) = \frac{b^2}{a}.$$

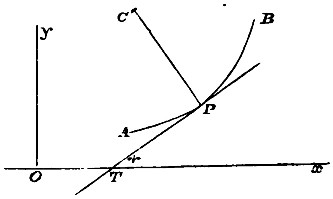
Similarly it may be shown that the radius of curvature at an extremity of the minor axis is equal to $\frac{a}{b}$.

CHAPTER VI.

ANALYTICAL DETERMINATION OF THE CENTRE OF CURVATURE. THEORY OF EVOLUTES AND INVOLUTES.

Determination of the Coordinates of the Centre of Curvature.

135. Let α, β, be the coordinates of C, the centre of the osculating circle of a curve AB at the point P. Then, x, y,



being the coordinates of P, ψ the inclination of the tangent PT at P to the axis of x, it is plain that

$$\alpha - x = -\rho \sin\psi = -\frac{ds}{d\psi}\sin\psi,$$

and $$\beta - y = \rho\cos\psi = \frac{ds}{d\psi}\cos\psi.$$

But $$\cos\psi = \frac{dx}{ds}, \quad \sin\psi = \frac{dy}{ds},$$

and $$d\psi = d\tan^{-1}\left(\frac{dy}{dx}\right) = \frac{dx\,d^2y - dy\,d^2x}{dx^2 + dy^2};$$

hence $$\alpha - x = -dy\,\frac{dx^2 + dy^2}{dx\,d^2y - dy\,d^2x},$$

and $$\beta - y = dx\,\frac{dx^2 + dy^2}{dx\,d^2y - dy\,d^2x}.$$

These two formulæ will enable us to determine α and β for any assigned point of the curve.

Suppose s to be the independent variable; then differentiating the equation
$$dx^2 + dy^2 = ds^2,$$
we have
$$dx\, d^2x + dy\, d^2y = 0,$$
and therefore the formulæ for α and β are reduced to

$$\alpha - x = d^2x \cdot \frac{dx^2 + dy^2}{(d^2x)^2 + (d^2y)^2},$$

$$\beta - y = d^2y \cdot \frac{dx^2 + dy^2}{(d^2x)^2 + (d^2y)^2}.$$

Formulæ for the Coordinates of the Centre of Curvature in terms of Partial Differential Coefficients of u.

136. From Art. (135) we see that
$$(\alpha - x)\, dx + (\beta - y)\, dy = 0 \ldots\ldots\ldots\ldots (1),$$
and
$$(\alpha - x)\, d^2x + (\beta - y)\, d^2y - dx^2 - dy^2 = 0 \ldots\ldots (2).$$

Between these two equations, together with the equation to the curve and its first and second differentials, viz.
$$u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$\frac{du}{dx}\, dx + \frac{du}{dy}\, dy = 0 \ldots\ldots\ldots\ldots\ldots (4),$$
$$\frac{du}{dx}\, d^2x + \frac{du}{dy}\, d^2y + \frac{d^2u}{dx^2}\, dx^2 + 2\,\frac{d^2u}{dx\,dy}\, dx\, dy + \frac{d^2u}{dy^2}\, dy^2 = 0 \ldots (5),$$
we may eliminate the six quantities x, y, dx, dy, d^2x, d^2y. In fact, from (1) and (4) we have
$$(\alpha - x)\,\frac{du}{dy} = (\beta - y)\,\frac{du}{dx} \ldots\ldots\ldots\ldots (6),$$
and therefore, from (2),
$$\frac{(\alpha - x)}{\dfrac{du}{dx}} \left(\frac{du}{dx}\, d^2x + \frac{du}{dy}\, d^2y\right) = dx^2 + dy^2,$$

whence, by (5), we have

$$dx^2 + dy^2 + \frac{(\alpha - x)}{\frac{du}{dx}}\left(\frac{d^2u}{dx^2}\,dx^2 + 2\,\frac{d^2u}{dx\,dy}\,dx\,dy + \frac{d^2u}{dy^2}\,dy^2\right) = 0,$$

and therefore, by (4) and (6),

$$\frac{\alpha - x}{\frac{du}{dx}} = \frac{\beta - y}{\frac{du}{dy}} = -\frac{\frac{du^2}{dx^2} + \frac{du^2}{dy^2}}{\frac{du^2}{dy^2}\,\frac{d^2u}{dx^2} - 2\,\frac{du}{dx}\,\frac{du}{dy}\,\frac{d^2u}{dx\,dy} + \frac{du^2}{dx^2}\,\frac{d^2u}{dy^2}} \;\ldots\; (7).$$

Locus of the Centre of Curvature.

137. Between the equation $u = 0$ and the equations (7) of the preceding Article, we may eliminate x and y: we shall thus obtain an equation between α and β alone, which will be the equation to the geometrical locus of C, the centre of curvature, when the point P of the curve is supposed to be variable in position. The locus of C, for a reason shortly to be explained, is called the *evolute* of the curve AB, which is itself called the *involute*.

Ex. To find the equation to the locus of the centre of curvature of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Here $$u = \tfrac{1}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0,$$

$$\frac{du}{dx} = \frac{x}{a^2},\quad \frac{du}{dy} = \frac{y}{b^2},$$

$$\frac{d^2u}{dx^2} = \frac{1}{a^2},\quad \frac{d^2u}{dx\,dy} = 0,\quad \frac{d^2u}{dy^2} = \frac{1}{b^2},$$

$$\frac{du^2}{dy^2}\,\frac{d^2u}{dx^2} - 2\,\frac{du}{dx}\,\frac{du}{dy}\,\frac{d^2u}{dx\,dy} + \frac{du^2}{dx^2}\,\frac{d^2u}{dy^2} = \frac{1}{a^2b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{1}{a^2b^2};$$

hence $$(\alpha - x)\,\frac{a^2}{x} = (\beta - y)\,\frac{b^2}{y} = -a^2b^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$

$$= -a^2 + \frac{a^2 - b^2}{a^2}\,x^2 = -b^2 + \frac{b^2 - a^2}{b^2}\,y^2,$$

and therefore

$$a^2\frac{\alpha}{x} = \frac{a^2 - b^2}{a^2}x^2, \quad b^2\frac{\beta}{y} = \frac{b^2 - a^2}{b^2}y^2,$$

or

$$\frac{x}{a} = \left(\frac{a\alpha}{a^2 - b^2}\right)^{\frac{1}{3}}, \quad \frac{y}{b} = -\left(\frac{b\beta}{a^2 - b^2}\right)^{\frac{1}{3}}.$$

Substituting these values of $\frac{x}{a}$, $\frac{y}{b}$, in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we get, for the equation to the evolute,

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

To shew that the Normal at any point of the Involute is a Tangent at the corresponding point of the Evolute.

138. By the formulæ (7) of Art. (136) and the equation $u = 0$, we may obtain α and β in terms of x; thus α and β, as well as y, are functions of x: hence as x varies, α and β as well as y must simultaneously vary. Differentiating the equation

$$(\alpha - x)\,dx + (\beta - y)\,dy = 0,$$

we shall accordingly obtain

$$(\alpha - x)\,d^2x + (\beta - y)\,d^2y + d\alpha\,dx + d\beta\,dy - dx^2 - dy^2 = 0,$$

and therefore, by virtue of the equation

$$(\alpha - x)\,d^2x + (\beta - y)\,d^2y - dx^2 - dy^2 = 0,$$

we have

$$d\alpha\,dx + d\beta\,dy = 0.$$

This equation shews that the tangent to the involute at the point (x, y) is at right angles to the tangent to the evolute at the corresponding point (α, β). Hence the normal at (x, y), which passes through C, must be a tangent to the evolute at (α, β).

Generation of the Involute by the end of a thread unwound from the Evolute.

139. Since

$$\alpha - x = -\rho \sin\psi, \qquad \beta - y = \rho \cos\psi,$$

we have $$(\alpha - x)^2 + (\beta - y)^2 = \rho^2 \ldots\ldots\ldots\ldots (1),$$

and therefore

$$(\alpha - x)(d\alpha - dx) + (\beta - y)(d\beta - dy) = \rho\, d\rho \ldots\ldots (2).$$

Also we know that

$$(\alpha - x)\, dx + (\beta - y)\, dy = 0 \ldots\ldots\ldots\ldots (3),$$

and $$d\alpha\, dx + d\beta\, dy = 0 \ldots\ldots\ldots\ldots\ldots (4).$$

From (2) and (3) we see that

$$(\alpha - x)\, d\alpha + (\beta - y)\, d\beta = \rho\, d\rho \ldots\ldots\ldots (5),$$

and, from (3) and (4),

$$(\alpha - x)\, d\beta - (\beta - y)\, d\alpha = 0 \ldots\ldots\ldots\ldots (6).$$

Adding together the squares of (5) and (6), we obtain

$$\{(\alpha - x)^2 + (\beta - y)^2\}(d\alpha^2 + d\beta^2) = \rho^2\, d\rho^2,$$

and therefore, by (1),

$$d\alpha^2 + d\beta^2 = d\rho^2.$$

Let σ denote an arc of the evolute originating at any proposed point and terminating at (α, β): then

$$d\sigma^2 = d\alpha^2 + d\beta^2 = d\rho^2;$$

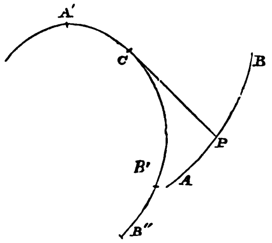
and therefore, the positive or negative sign being chosen accordingly as ρ decreases or increases with the increase of σ,

$$d\sigma \pm d\rho = 0, \qquad \sigma \pm \rho = c,$$

c being some constant quantity.

We proceed now to give the geometrical interpretation of this result.

First take $\sigma + \rho = c$. Let C and P be any two corresponding points in the evolute and involute respectively. Let $A'CB'$ be an arc of the evolute. Join CP, which will be the radius of curvature to the involute at the point *P*, and a tangent to the evolute at the point *C*. Let $A'C = \sigma$, and $A'B' = c$. Then



$$A'C + CP = c = A'B'.$$

Hence it is obvious that, if a thread, of which the length is *c*, be fixed with one end at *A'*, so as to touch the curve at this point, and be wound about the curve *A'B'* by a hand taking hold of the string at *P*, its extremity *P* will trace out the involute *AB*.

Next take $\sigma - \rho = c$. Let $B''B' = c$, $B''C = \sigma$, the origin of the arcs being now some fixed point *B''*. Then

$$B''C - CP = c = B''B',$$

a result which points out the very same geometrical property as when we adopted the positive sign. From the geometrical property which we have established have arisen the names *evolute* and *involute*.

To find the length of any Arc of the Evolute of a Curve.

140. By the preceding Article we know that

$$\sigma \pm \rho = c.$$

Let σ_1, ρ_1, and σ_2, ρ_2, be corresponding values of σ, ρ: then

$$\sigma_1 \pm \rho_1 = c = \sigma_2 \pm \rho_2,$$

or

$$\sigma_1 \sim \sigma_2 = \rho_1 \sim \rho_2.$$

Hence, to find the length of an arc of the evolute corresponding to any proposed arc of the involute, we must take the difference between the radii of curvature at the two extremities of the latter arc, and this will be the length of the former; provided that, for the whole interval, ρ either always decreases or always increases as σ increases.

Ex. To find the length of the whole evolute of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The radii of curvature at the extremities of the axes major and minor are $\frac{b^2}{a}$ and $\frac{a^2}{b}$; hence the length of a quarter of the whole evolute is equal to

$$\frac{a^2}{b} - \frac{b^2}{a},$$

or the length of the whole evolute is equal to

$$4\,\frac{a^3 - b^3}{ab}.$$

CHAPTER VII.

CONTACT OF CURVES.

Definition of Order of Contact.

141. Let $y' = f(x')$, $y' = F(x')$, be the equations to two curves. Suppose (x, y) to be a point common to both curves.

The two curves are said to have a contact of the first order at the point (x, y), if

$$F'(x) = f'(x);$$

of the second order, if

$$F'(x) = f'(x), \quad F''(x) = f''(x);$$

of the third order, if

$$F'(x) = f'(x), \quad F''(x) = f''(x), \quad F'''(x) = f'''(x),$$

and so on; the contact being of the r^{th} order if

$$F'(x) = f'(x), \quad F''(x) = f''(x), \quad F'''(x) = f'''(x), \ldots F^{r}(x) = f^{r}(x).$$

The higher the order of Contact, the closer the Contact.

142. Let the curve $y' = \phi(x')$ have, at the point (x, y), a contact of the m^{th} order with the curve $y' = F(x')$, and of the n^{th} order with the curve $y' = f(x')$, and suppose m to be greater than n. Then, by the theory of vanishing fractions, when h becomes less than any assignable magnitude,

$$\frac{F(x+h) - \phi(x+h)}{f(x+h) - \phi(x+h)} = \frac{F'(x+h) - \phi'(x+h)}{f'(x+h) - \phi'(x+h)}$$

$$= \frac{F''(x+h) - \phi''(x+h)}{f''(x+h) - \phi''(x+h)} = \&\text{c.}$$

$$= \frac{F^{n}(x+h) - \phi^{n}(x+h)}{f^{n}(x+h) - \phi^{n}(x+h)} = \frac{0}{f^{n}(x) - \phi^{n}(x)} = 0,$$

or, corresponding to a small increment h of x, the difference between the ordinates of the curves $y' = F(x')$, $y' = \phi(x')$, is indefinitely small in comparison with the difference between the ordinates of the curves $y' = f(x')$, $y' = \phi(x')$. Thus the contact is infinitely closer when of the m^{th} than when of the n^{th} order.

Order of Contact dependent upon the number of Parameters.

143. Let the general equation to a family of curves be

$$u' = 0 \text{ .... } (1),$$

u' being a function of x', y', the coordinates of any point whatever in any one of the curves, and of n parameters $a_1, a_2, a_3, \ldots. a_n$. Differentiating the equation to the curve $n-1$ times successively with regard to x' as the independent variable, we get

$$\left.\begin{array}{l} \dfrac{Du'}{dx'} = 0, \\ \dfrac{D^2u'}{dx'^2} = 0, \\ \dfrac{D^3u'}{dx'^3} = 0, \\ \ \ \cdot \quad \cdot \quad \cdot \\ \ \ \cdot \quad \cdot \quad \cdot \\ \dfrac{D^{n-1}u'}{dx'^{n-1}} = 0, \end{array}\right\} \text{ .... } (2),$$

$n - 1$ equations involving x', y', and the $n - 1$ differential coefficients

$$\frac{dy'}{dx'}, \ \frac{d^2y'}{dx'^2}, \ \frac{d^3y'}{dx'^3}, \ldots. \ \frac{d^{n-1}y'}{dx'^{n-1}}.$$

Since we have n equations, (1) and (2), involving n parameters and $n + 1$ quantities

$$\left(x', y', \frac{dy'}{dx'}, \frac{d^2y'}{dx'^2}, \frac{d^3y'}{dx'^3}, \ldots. \frac{d^{n-1}y'}{dx'^{n-1}}\right) \text{......} (3),$$

it is evident that we may assign any values we please to these $n + 1$ quantities, the values of the n parameters being determined accordingly. We may therefore obtain the equation to an individual of the family of curves denoted by the equation (1) which shall have a contact of the $(n-1)^{\text{th}}$ order with any proposed curve at any point (x, y), by assigning to the quantities (3) the values of the corresponding quantities in the proposed curve, and obtaining the values of the n parameters accordingly. Thus suppose that $u' = F(x', y', a_1, a_2, a_3, \ldots\ldots a_n)$ and that $f(x)$ is the ordinate in the proposed curve at the point of contact: then $v_1, v_2, v_3, \ldots\ldots v_n$, denoting certain functions of x, we shall have

$$a_1 = v_1,\ a_2 = v_2,\ a_3 = v_3,\ \ldots\ldots\ a_n = v_n,$$

and the equation

$$F(x', y', v_1, v_2, v_3, \ldots\ldots v_n) = 0,$$

will represent an individual of the family of curves represented by the equation (1), which shall have a contact of the $(n-1)^{\text{th}}$ order with the proposed curve at the point (x, y).

Ex. 1. The general equation to a circle is

$$(x' - a)^2 + (y' - \beta)^2 = \rho^2,$$

a, β, ρ, being disposable constants, upon the particular magnitudes of which the dimensions and position of the circle depend.

Let it be proposed to determine the values of a, β, ρ, that the circle may have a contact of the second order with any proposed curve $y'' = \phi(x'')$ at a point (x, y) of the curve.

Differentiating the equation to the circle twice with regard to x', we have

$$x' - a + (y' - \beta)\frac{dy'}{dx'} = 0,$$

$$1 + (y' - \beta)\frac{d^2y'}{dx'^2} + \frac{dy'^2}{dx'^2} = 0.$$

If then for $$x',\ y',\ \frac{dy'}{dx'},\ \frac{d^2y'}{dx'^2},$$

we substitute the quantities

$$x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},$$

where $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are supposed to denote the values of $\frac{dy''}{dx''}$ and $\frac{d^2y''}{dx''^2}$ at the point (x, y) of the curve, we shall have for the determination of α, β, ρ, the three equations

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2,$$

$$x - \alpha + (y - \beta)\frac{dy}{dx} = 0,$$

$$1 + (y - \beta)\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} = 0.$$

Now the equations (1) and (2) of Art. (136), supposing d^2x to be zero, or x to be the independent variable, coincide with the last two of these equations. We see therefore that the coordinates of the centre of the circle, which has a contact of the second order with any proposed curve at any proposed point, coincide with those of the centre of the osculating circle at the same point; or that the osculating circle is identical with the circle which has a contact of the second order.

Ex. 2. To determine the parabola which has a contact of the second order with an ellipse at an extremity of the latus rectum of the ellipse; the equation to the ellipse being

$$\frac{x''^2}{a^2} + \frac{2y''^2}{a^2} = 1,$$

and the axis of the parabola being parallel to the major axis of the ellipse.

Let the equation to the parabola be

$$(y' + \beta)^2 = 4m(x' + \alpha) \ldots\ldots\ldots\ldots (1);$$

then

$$(y + \beta)\frac{dy}{dx} = 2m \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{dy^2}{dx^2} + (y + \beta)\frac{d^2y}{dx^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From the equation to the ellipse there is

$$x + 2y\frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$1 + 2\,\frac{dy^2}{dx^2} + 2y\,\frac{d^2y}{dx^2} = 0 \ldots\ldots\ldots\ldots\ldots \text{(5)}.$$

Now the coordinates of an extremity of the latus rectum of the ellipse are $\frac{a}{\sqrt{2}}, \frac{a}{2}$: hence, from (4),

$$\frac{a}{\sqrt{2}} + a\,\frac{dy}{dx} = 0,\quad \frac{dy}{dx} = -\frac{1}{\sqrt{2}},$$

and therefore, from (5),

$$2 + a\,\frac{d^2y}{dx^2} = 0,\quad \frac{d^2y}{dx^2} = -\frac{2}{a}.$$

Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (3), we get

$$\tfrac{1}{2} + (\tfrac{1}{2}a + \beta)\,.\,\frac{-2}{a} = 0,\quad \beta = -\tfrac{1}{4}a;$$

hence also, from (2),

$$(\tfrac{1}{2}a - \tfrac{1}{4}a)\,.\,\frac{-1}{\sqrt{2}} = 2m,\quad 4m = -\frac{a}{2\sqrt{2}},$$

and thence, by (1),

$$(\tfrac{1}{2}a - \tfrac{1}{4}a)^2 = -\frac{a}{2\sqrt{2}}\left(\frac{a}{\sqrt{2}} + \alpha\right),$$

$$\frac{a}{\sqrt{2}} + \alpha = -\frac{a\sqrt{2}}{8},\quad \alpha = -\frac{5a}{4\sqrt{2}}.$$

Hence the equation to the required parabola will be

$$\left(y' - \frac{a}{4}\right)^2 = -\frac{a}{2\sqrt{2}}\left(x' - \frac{5a}{4\sqrt{2}}\right).$$

When the Radius of Curvature is a Maximum or Minimum, the Contact is of the third order.

144. Let the equation to the curve be

$$y'' = \phi(x''),$$

and the equation to the circle of curvature, at a point (x, y) of the curve,

$$(x' - \alpha)^2 + (y' - \beta)^2 = \rho^2 \ldots\ldots\ldots\ldots \text{(1)}.$$

Then, putting $\frac{dy'}{dx'} = p'$, $\frac{d^2y'}{dx'^2} = q'$, $\frac{d^3y'}{dx'^3} = r'$, we may get from (1), by two differentiations,

$$\rho^2 = \frac{(1 + p'^2)^3}{q'^2},$$

$$2 \log \rho = 3 \log (1 + p'^2) - 2 \log q',$$

and therefore, differentiating again, ρ being invariable as we pass from one point of the circle to another,

$$0 = \frac{3p'q'}{1 + p'^2} - \frac{r'}{q'},$$

or

$$r' = \frac{3p'q'^2}{1 + p'^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Again, putting p, q, r, for the values of $\frac{dy''}{dx''}$, $\frac{d^2y''}{dx''^2}$, $\frac{d^3y''}{dx''^3}$, at the point (x, y), we know that

$$\rho^2 = \frac{(1 + p^2)^3}{q^2}, \quad 2 \log \rho = 3 \log (1 + p^2) - 2 \log q.$$

But, if ρ be a maximum or minimum, $\frac{d\rho}{dx} = 0$, and therefore at such a point we have

$$0 = \frac{3pq}{1 + p^2} - \frac{r}{q}, \quad r = \frac{3pq^2}{1 + p^2} \ldots\ldots\ldots\ldots \quad (3).$$

Now, by the nature of the contact between a curve and its circle of curvature, the values of p', q', at the point (x, y), are p, q; hence, by (2) and (3), we see that the values of r' at the point (x, y) is the same as that of r. Thus the contact between the circle and the curve must be of the third order.

CHAPTER VIII.

ENVELOPS.

Case of a single Parameter.

145. LET the equation to a family of curves be

$$f(x', y', a) = 0 \quad \ldots\ldots\ldots\ldots\ldots (1),$$

x', y', being the coordinates of any point in any one of the curves, and a being a parameter, the particular values of which determine the individual curves of the family. Suppose that a becomes $a + \delta a$, δa being an indefinitely small increment of a. Then the equation (1) becomes

$$f(x', y', a + \delta a) = 0 \quad \ldots\ldots\ldots\ldots (2).$$

Let x, y, be the values of x', y', at the intersection of the curves (1) and (2); that is, of any two consecutive individuals of the family of curves. Then

$$f(x, y, a) = 0 \quad \ldots\ldots\ldots\ldots\ldots (3),$$

and $$f(x, y, a + \delta a) = 0.$$

Hence $$\frac{f(x, y, a + \delta a) - f(x, y, a)}{\delta a} = 0,$$

and therefore, when δa is diminished without limit, we have ultimately

$$\frac{df(x, y, a)}{da} = 0 \quad \ldots\ldots\ldots\ldots\ldots (4).$$

Between the two equations (3) and (4) we may eliminate the parameter a, and we shall thus obtain an equation

$$\phi(x, y) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

expressing the relation between the coordinates of the point of intersection of any and every two consecutive individuals of the family of curves: this equation will therefore represent a curve, which is the locus of such consecutive intersections.

It is easy to see that the curve (5) touches each of the individuals of the family (1). In fact, differentiating the equation (1), we get

$$\frac{d}{dx'}f(x', y', a)\,dx' + \frac{d}{dy'}f(x', y', a)\,dy' = 0 \;\ldots\ldots (6).$$

Again, since, by virtue of (3) and (4), a as well as y is a function of x, we have, differentiating (3),

$$\frac{d}{dx}f(x, y, a)\,dx + \frac{d}{dy}f(x, y, a).\,dy + \frac{d}{da}f(x, y, a)\,da = 0,$$

or, by the aid of (4),

$$\frac{d}{dx}f(x, y, a)\,dx + \frac{d}{dy}f(x, y, a)\,dy = 0 \;\ldots\ldots (7).$$

Now, when x', y', are replaced by x, y,

$$\frac{d}{dx'}f(x', y', a) = \frac{d}{dx}f(x, y, a),$$

and

$$\frac{d}{dy'}f(x', y', a) = \frac{d}{dy}f(x, y, a);$$

hence, from (6) and (7), we see that the ratio of dy' to dx' is the same in the curve (1) as that of dy to dx in the curve (5) at their common point (x, y).

The locus of the consecutive intersections of the individuals of a family of curves has been called their *envelop* in consequence of this property.

Ex. 1. To find the nature of the curve which shall touch all the curves represented by the equation

$$y = ax - a^2,$$

whatever be the value of a.

Differentiating with regard to a, we have

$$0 = x - 2a;$$

and therefore, eliminating a between these two equations, we obtain, for the equation to the required envelop,

$$4y = x^2,$$

which represents a parabola.

Ex. 2. Straight lines are drawn at right angles to the tangents of a parabola at the points where they meet a given straight line perpendicular to the axis: to find the envelop of these straight lines.

The equation to a tangent to the parabola $y^2 = 4mx$ will be

$$y = ax + \frac{m}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1):$$

let that to the given line be

$$x + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The coordinates of the intersection of (1) and (2) will be

$$y = -ca + \frac{m}{a},$$
$$x = -c:$$

hence the equation to the perpendicular to the tangent will be

$$y + ca - \frac{m}{a} = -\frac{1}{a}(x + c),$$

or
$$ay - m + ca^2 = -(x + c) \ldots\ldots\ldots\ldots\ldots (3).$$

Differentiating (3) with respect to the variable parameter a, we get

$$y + 2ca = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Eliminating a between (3) and (4) we have, for the envelop of (3), the parabola

$$y^2 = 4c\,\{x - (m - c)\} \quad \ldots\ldots\ldots\ldots \quad (5).$$

From the form of this equation it appears that, to arrive at its vertex we must proceed from the origin for a space $m - c$ along the axis of x, and to arrive at the focus we must afterwards proceed for a space c: hence we shall have proceeded in all for a space m to arrive at the focus. Thus (5) represents a parabola confocal with $y^2 = 4mx$.

General case of any number of Parameters.

146. Let the equation to a family of curves be

$$u' = f(x', y', a_1, a_2, a_3, \ldots a_n) = 0 \ldots\ldots (1),$$

x', y', being the coordinates of any point in any one of the curves, and a_1, a_2, a_3, ... a_n, being n parameters, the particular values of which determine the individual curves of the family. We suppose these n parameters to be connected together by $n - 1$ equations, so that any $n - 1$ of them will be functional of the n^{th} remaining one. Let the $n - 1$ equations be

$$(\phi_1 = 0,\ \phi_2 = 0,\ \phi_3 = 0,\ \ldots\ \phi_{n-1} = 0) \ldots\ldots (2).$$

Conceive a_1, a_2, a_3, ... a_n, to become $a_1 + \delta a_1$, $a_2 + \delta a_2$, $a_3 + \delta a_3$, ... $a_n + \delta a_n$; δa_1, δa_2, δa_3, ... δa_n, being indefinitely small increments of the n parameters consistent with the simultaneous equations (2). Then the equation (1) becomes

$$u' + \delta u' = f(x', y', a_1 + \delta a_1, a_2 + \delta a_2, a_3 + \delta a_3, \ldots a_n + \delta a_n) = 0 \ldots (3).$$

Let x, y, be the values of x', y', at the intersection of the curves (1) and (3), that is, of any two consecutive individuals of the family of curves. Then, putting for the sake of brevity, a'_1, a'_2, a'_3, ... a'_n, in place of $a_1 + \delta a_1$, $a_2 + \delta a_2$, $a_3 + \delta a_3$, ... $a_n + \delta a_n$, we have, u and $u + \delta u$ representing the values of u' and $u' + \delta u'$, when x', y', are replaced by x, y,

$$u = f(x, y, a_1, a_2, a_3, \ldots a_n) = 0 \ldots\ldots\ldots (4),$$

and $\quad u + \delta u = f(x, y, a'_1, a'_2, a'_3, \ldots a'_n) = 0$:

from these two equations we see that

$$\frac{\Delta u}{\delta a_1} = 0,$$

the numerator of the left-hand member of this equation denoting the total increment of the function u due directly and indirectly to an increment δa_1 of a_1. Hence, proceeding to the limit, when δa_1 becomes less than any assignable magnitude, we have

$$\frac{Du}{da_1} = 0, \quad \text{or} \quad Du = 0;$$

or, expressing the total in terms of the partial differentials,

$$\frac{du}{da_1}da_1 + \frac{du}{da_2}da_2 + \frac{du}{da_3}da_3 + \dots + \frac{du}{da_n}da_n = 0 \dots . (5).$$

Differentiating the $n-1$ equations (2), we get also the $n-1$ differential equations

$$\left.\begin{array}{l}
\frac{d\phi_1}{da_1}da_1 + \frac{d\phi_1}{da_2}da_2 + \frac{d\phi_1}{da_3}da_3 + \dots + \frac{d\phi_1}{da_n}da_n = 0 \\
\frac{d\phi_2}{da_1}da_1 + \frac{d\phi_2}{da_2}da_2 + \frac{d\phi_2}{da_3}da_3 + \dots + \frac{d\phi_2}{da_n}da_n = 0 \\
\frac{d\phi_3}{da_1}da_1 + \frac{d\phi_3}{da_2}da_2 + \frac{d\phi_3}{da_3}da_3 + \dots + \frac{d\phi_3}{da_n}da_n = 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\frac{d\phi_{n-1}}{da_1}da_1 + \frac{d\phi_{n-1}}{da_2}da_2 + \frac{d\phi_{n-1}}{da_3}da_3 + \dots + \frac{d\phi_{n-1}}{da_n}da_n = 0
\end{array}\right\} \dots (6).$$

Adding the equations (6) multiplied in order by $(n-1)$ arbitrary quantities $\lambda_1, \lambda_2, \lambda_3, \dots . \lambda_{n-1}$, to the equation (5), and equating to zero the coefficients of the differentials of the resulting equation, we obtain the n equations

$$\left.\begin{array}{l}
\frac{du}{da_1} + \lambda_1\frac{d\phi_1}{da_1} + \lambda_2\frac{d\phi_2}{da_1} + \lambda_3\frac{d\phi_3}{da_1} + \dots + \lambda_{n-1}\frac{d\phi_{n-1}}{da_1} = 0 \\
\frac{du}{da_2} + \lambda_1\frac{d\phi_1}{da_2} + \lambda_2\frac{d\phi_2}{da_2} + \lambda_3\frac{d\phi_3}{da_2} + \dots + \lambda_{n-1}\frac{d\phi_{n-1}}{da_2} = 0 \\
\frac{du}{da_3} + \lambda_1\frac{d\phi_1}{da_3} + \lambda_2\frac{d\phi_2}{da_3} + \lambda_3\frac{d\phi_3}{da_3} + \dots + \lambda_{n-1}\frac{d\phi_{n-1}}{da_3} = 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\frac{du}{da_n} + \lambda_1\frac{d\phi_1}{da_n} + \lambda_2\frac{d\phi_2}{da_n} + \lambda_3\frac{d\phi_3}{da_n} + \dots + \lambda_{n-1}\frac{d\phi_{n-1}}{da_n} = 0
\end{array}\right\} \dots (7).$$

Between the equations (2), (4), (7), $2n$ in number, we may eliminate the $2n-1$ quantities $a_1, a_2, a_3, \dots a_n, \lambda_1, \lambda_2, \lambda_3, \dots \lambda_{n-1}$, and we shall then arrive at an equation

$$F(x, y) = 0$$

representing the required envelop of the family of curves.

Ex. 1. To find the equation to a curve which shall touch each of the family of straight lines defined by the equation

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1 \;\ldots\ldots\ldots\ldots\ldots (1),$$

α and β being connected together by the equation

$$\alpha^n + \beta^n = c^n \;\ldots\ldots\ldots\ldots\ldots (2).$$

Differentiating (1) and (2) with regard to α and β, we have

$$\frac{x\, d\alpha}{\alpha^2} + \frac{y\, d\beta}{\beta^2} = 0 \;\ldots\ldots\ldots\ldots (3),$$

$$\alpha^{n-1} d\alpha + \beta^{n-1} d\beta = 0 \;\ldots\ldots\ldots\ldots (4).$$

From (3) and (4) there is

$$\left.\begin{aligned} \frac{\lambda x}{\alpha^2} &= \alpha^{n-1}, \\ \frac{\lambda y}{\beta^2} &= \beta^{n-1}, \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots (5).$$

Multiplying the two equations (5) by α, β, adding and attending to (1) and (2), we see that

$$\lambda = c^n:$$

hence, from (5), $\alpha^{n+1} = c^n x, \quad \beta^{n+1} = c^n y,$

and therefore, from (2),

$$x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}},$$

the equation to the required envelop.

Ex. 2. An ellipse moves with its centre in the arc of an equal similar ellipse, and has its axes parallel to the axes of the fixed ellipse: to find the curve which envelops the moveable one.

Let a, b, be the semi-axes of either ellipse; the equation to the moveable ellipse will be

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1 \;\ldots\ldots\ldots\ldots (1),$$

where α, β, are subject to the condition

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \text{ .... } (2).$$

Differentiating with regard to the parameters, we have

$$\frac{x-\alpha}{a^2}\,d\alpha+\frac{y-\beta}{b^2}\,d\beta=0,$$

$$\frac{\alpha\,d\alpha}{a^2}+\frac{\beta\,d\beta}{b^2}=0,$$

and therefore, λ being an indeterminate multiplier,

$$\left.\begin{aligned}\frac{\lambda\alpha}{a^2}&=\frac{x-\alpha}{a^2}\\ \frac{\lambda\beta}{b^2}&=\frac{y-\beta}{b^2}\end{aligned}\right\} \text{ .... } (3).$$

By (1), (2), (3), we have

$$\lambda=\frac{\alpha(x-\alpha)}{a^2}+\frac{\beta(y-\beta)}{b^2},$$

and $$\lambda\left\{\frac{\alpha(x-\alpha)}{a^2}+\frac{\beta(y-\beta)}{b^2}\right\}=1,$$

and therefore $$\lambda^2=1,\quad \lambda=\pm 1.$$

If $\lambda=1$, we have, from the equations (3),

$$\alpha=x-\alpha,\quad \alpha=\tfrac{1}{2}x,\quad \beta=\tfrac{1}{2}y;$$

and therefore, from (2), we have for the equation to the envelop

$$\frac{x^2}{4a^2}+\frac{y^2}{4b^2}=1,$$

which is the equation to an ellipse similar in form to either of the original ones, its axes being of twice the magnitude.

Again, if $\lambda=-1$, we have from (3),

$$x=0,\quad y=0,$$

which are the equations to a point, viz. the centre of the fixed ellipse, through which it is evident that all the moveable ellipses pass.

Intersection of Consecutive Normals to a Curve.

147. The equation to the normal at any point x, y, of a curve $f(x', y') = 0$, will be

$$(x'-x)\frac{du}{dy} = (y'-y)\frac{du}{dx} \ldots\ldots\ldots\ldots (1),$$

where $u = f(x, y)$. Differentiating (1), considering x as a variable parameter, of which y, $\frac{du}{dx}$, and $\frac{du}{dy}$ are functional, we get

$$(x'-x)\left(\frac{d^2u}{dx\,dy}dx + \frac{d^2u}{dy^2}dy\right) - \frac{du}{dy}dx$$

$$= (y'-y)\left(\frac{d^2u}{dx^2}dx + \frac{d^2u}{dx\,dy}dy\right) - \frac{du}{dx}dy \ldots\ldots (2).$$

From (1) and (2), and the differential of the equation $u = 0$, we see that

$$\frac{x'-x}{\frac{du}{dx}} = \frac{y'-y}{\frac{du}{dy}} = -\frac{\frac{du^2}{dx^2} + \frac{du^2}{dy^2}}{\frac{du^2}{dy^2}\frac{d^2u}{dx^2} - 2\frac{du}{dx}\frac{du}{dy}\frac{d^2u}{dx\,dy} + \frac{du^2}{dx^2}\frac{d^2u}{dy^2}},$$

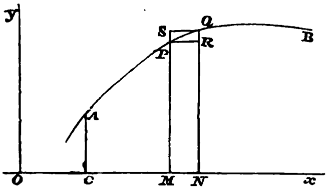
the values of x', y', the coordinates of the intersection of two consecutive normals, coinciding with their values obtained otherwise in Art. (146).

CHAPTER IX.

DIFFERENTIALS OF AREAS, VOLUMES, ARCS, AND SURFACES

Differential of an Area.

148. Let AB be any portion of a curve referred to rectangular axes Ox, Oy. Let PM, QN, be the ordinates of two points P, Q, of the curve, very near to each other. Let AC be the ordinates of A. Draw PR and QS, at right angles to QN and MP produced. Let $OM = x$, $PM = y$, $ON = x + \delta x$, $QN = y + \delta y$; let $A =$ the area $ACMP$, and $A + \delta A =$ the area $ACNQ$.



Then, since the area $PQNM$ is evidently less than $SQNM$ and greater than $PRNM$, it is plain that

$$y\delta x < \delta A < (y + \delta y)\,\delta x,$$

or

$$y < \frac{\delta A}{\delta x} < y + \delta y.$$

Proceeding to the limit, when δx and δy become less than any assignable magnitudes, $y + \delta y = y$: hence ultimately, replacing small increments by differentials, we see that

$$\frac{dA}{dx} = y, \quad \text{or } dA = y\,dx$$

Differential of a Volume of Revolution.

149. Conceive a surface to be generated by the complete revolution of the curve AB, in the diagram of the preceding Article, about the axis of x. Let V represent the volume generated by the area $ACMP$, and $V + \delta V$ that generated by the area $ACNQ$. Now it is shewn in ordinary treatises on Trigonometry that the area of a circle is equal to $\pi \cdot (\text{radius})^2$, where π is the circular measure of 180°: hence the areas of the circles generated by the revolution of the ordinates PM, QN, will be equal to

$$\pi y^2, \quad \pi (y + \delta y)^2,$$

and therefore the volumes of the thin cylinders generated by the revolution of the areas $PRNM$, $SQNM$, will be equal to

$$\pi y^2 \delta x, \quad \pi (y + \delta y)^2 \delta x.$$

But it is plain that δV is greater than the former and less than the latter of these cylinders: hence

$$\pi y^2 \delta x < \delta V < \pi (y + \delta y)^2 \delta x,$$

or

$$\pi y^2 < \frac{\delta V}{\delta x} < \pi (y + \delta y)^2.$$

Now ultimately, when δx and δy become less than any assignable magnitudes, πy^2 and $\pi (y + \delta y)^2$, the limits of the value of $\frac{\delta V}{\delta x}$, become equal to each other: hence, replacing indefinitely small increments by differentials, we have

$$\frac{dV}{dx} = \pi y^2,$$

or

$$dV = \pi y^2 dx.$$

Differential of an Arc.

150. Let the chord of the arc PQ in the figure of Art. (148) be denoted by c; let arc $AP = s$, arc $PQ = \delta s$. Then

$$c^2 = \delta x^2 + \delta y^2,$$

$$\frac{c^2}{\delta s^2} \frac{\delta s^2}{\delta x^2} = 1 + \frac{\delta y^2}{\delta x^2}.$$

Now ultimately, when δx is diminished without limit, we know by the 7th Lemma of Newton's *Principia* that $\frac{c}{\delta s} = 1$: hence, replacing infinitesimal differences by differentials, we have

$$\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2},$$

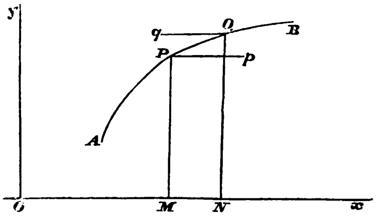
or

$$ds^2 = dx^2 + dy^2,$$

a relation which has been already established in Art. (100).

Differential of a Surface of Revolution.

151. Let S, $S + \delta S$, denote the areas of the surfaces generated by the revolution of the arcs AP, AQ, of the curve AB, about



the axis Ox; then δS will represent the area of the surface generated by the arc PQ. From P, Q, draw Pp, Qq, parallel to the axis of x and each equal to the arc PQ.

Let $AP = s$, $PQ = \delta s$. Then, if Pp, Qq, revolve about Ox together with the rest of the diagram, they will generate two thin cylinders. The length of each of these cylinders will be equal to the curvilinear distance between the circular ends of the surface generated by PQ; but the average radii of the circular sections of this surface will evidently be greater than those of the former and less than those of the latter cylinder. Hence it is manifest that, $2\pi y \delta s$ and $2\pi (y + \delta y) \delta s$ being the surfaces of the two cylinders,

$$2\pi y \delta s < \delta S < 2\pi (y + \delta y) \delta s,$$

or

$$2\pi y < \frac{\delta S}{\delta s} < 2\pi (y + \delta y):$$

but, when we proceed to the limit, by making δy less than any assignable magnitude, the two quantities $2\pi y$, $2\pi(y + \delta y)$, become equal to each other; hence, replacing indefinitely small differences by differentials, we see that

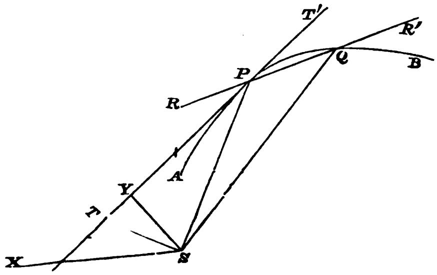
$$\frac{dS}{ds} = 2\pi y, \quad dS = 2\pi y ds.$$

CHAPTER X.

CURVES REFERRED TO POLAR COORDINATES.

Tangency.

152. In this chapter we shall investigate formulæ, in relation to curves referred to polar coordinates, analogous to those which in the preceding chapters have been established in regard to curves referred to rectilinear coordinates. We shall begin with the investigation of the formulæ of tangency.



Let P and Q be any two neighbouring points of a curve AB. Let S be the pole, and SX an indefinite straight line through S. Join SP, SQ, and let $SP = r$, $\angle PSX = \theta$, $SQ = r + \delta r$, $\angle QSX = \theta + \delta\theta$. Through P and Q draw the indefinite straight line RR' and let TT' be the tangent at P, which will be the ultimate position of the line RR' when the point Q approaches indefinitely near to P. From S draw SY, ST, at right angles to PT, PS, respectively: ST is called the *subtangent at the point P*. Let $SY = p$, $ST = v$, $\angle SPT = \phi$, $\angle SPR = \phi'$, arc $AP = s$, arc $PQ = \delta s$, chord $PQ = c$.

Then from the geometry it is plain that

$$c \sin \phi' = (r + \delta r) \sin \delta\theta,$$

or
$$\frac{c}{\delta s} \frac{\delta s}{\delta\theta} \sin \phi' = (r + \delta r) . \frac{\sin \delta\theta}{\delta\theta} \ldots\ldots\ldots\ldots (1);$$

and
$$c \cos \phi' + r = (r + \delta r) \cos \delta\theta,$$

or
$$\frac{c}{\delta s} \frac{\delta s}{\delta\theta} \cos \phi' + 2r \frac{\sin^2 \frac{\delta\theta}{2}}{\left(\frac{\delta\theta}{2}\right)^2} . \frac{\delta\theta}{4} = \frac{\delta r}{\delta\theta} \cos \delta\theta \ldots\ldots\ldots (2).$$

Now ultimately, when $\delta\theta$ becomes less than any assignable quantity,

$$\frac{c}{\delta s} = 1, \ \frac{\delta s}{\delta\theta} = \frac{ds}{d\theta}, \ \phi' = \phi, \ r + \delta r = r, \ \frac{\sin \delta\theta}{\delta\theta} = 1,$$

$$\frac{\sin \frac{\delta\theta}{2}}{\frac{\delta\theta}{2}} = 1, \ \cos \delta\theta = 1:$$

hence, from (1) and (2),

$$\frac{ds}{d\theta} \sin \phi = r,$$

or
$$\sin \phi = \frac{rd\theta}{ds} \ldots\ldots\ldots\ldots\ldots (3);$$

and
$$\frac{ds}{d\theta} \cos \phi = \frac{dr}{d\theta},$$

or
$$\cos \phi = \frac{dr}{ds} \ldots\ldots\ldots\ldots\ldots (4).$$

From (3) and (4), we have also

$$\tan \phi = \frac{rd\theta}{dr} \ldots\ldots\ldots\ldots\ldots (5).$$

Adding together the squares of (3) and (4), we get

$$1 = \frac{r^2 d\theta^2}{ds^2} + \frac{dr^2}{ds^2},$$

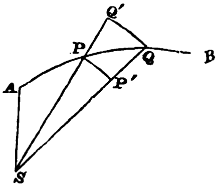
or
$$ds^2 = r^2 d\theta^2 + dr^2 \ldots\ldots\ldots\ldots\ldots (6).$$

Also $$SY = p = r \sin \phi = \frac{r^2 d\theta}{ds} \quad \ldots\ldots\ldots\ldots \quad (7),$$

and $$ST = v = r \tan \phi = \frac{r^2 d\theta}{dr} \quad \ldots\ldots\ldots\ldots \quad (8).$$

Differential of an Area.

153. Let *P* and *Q* be any two neighbouring points in a curve *AB*. Join *SA*, *SP*, *SQ*. With *S* as a centre describe two circular arcs, *PP′*, *QQ′*, cutting *SQ* and *SP* produced, respectively in *P′* and *Q′*. Let *A*, $A + \delta A$, denote the areas *ASP*, *ASQ*, respectively. Then it is evident that δA is intermediate in magnitude between the two circular sectors *SPP′*, *SQQ′*, that is,



$$\tfrac{1}{2} r^2 \delta\theta \angle \delta A \angle \tfrac{1}{2} (r + \delta r)^2 \delta\theta,$$

or $$\tfrac{1}{2} r^2 \angle \frac{\delta A}{\delta\theta} \angle \tfrac{1}{2} (r + \delta r)^2.$$

But ultimately, when $\delta\theta$, and therefore δr, becomes less than any assignable magnitude, the two quantities $\frac{1}{2} r^2$, $\frac{1}{2} (r + \delta r)^2$, assume a ratio of equality: hence, replacing $\frac{\delta A}{\delta\theta}$ by $\frac{dA}{d\theta}$, we have

$$\frac{dA}{d\theta} = \tfrac{1}{2} r^2,$$

or $$dA = \tfrac{1}{2} r^2 d\theta.$$

Cor. If *SA* be taken as the axis of *x*, and a perpendicular to *SA* through *S* as the axis of *y*, then

$$x = r \cos \theta, \quad y = r \sin \theta,$$

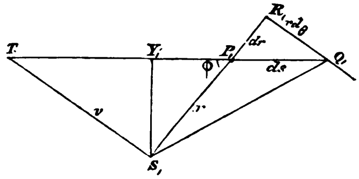
$$dx = dr \cos \theta - r \sin \theta \, d\theta, \quad dy = dr \sin \theta + r \cos \theta \, d\theta;$$

and therefore $$xdy - ydx = r^2 d\theta,$$

whence also $$dA = \tfrac{1}{2}(xdy - ydx).$$

Diagram of Differentials.

154. From any point S_1 draw any line S_1P_1 equal to r; produce S_1P_1 to a point R_1 such that $P_1R_1 = dr$. From the point R_1 draw R_1Q_1 at right angles to S_1R_1, and equal to $rd\theta$.



Join Q_1S_1, Q_1P_1, and from S_1 draw S_1Y_1, S_1T_1, to meet Q_1P_1 produced, in Y_1, T_1, the line S_1Y_1 being at right angles to P_1T_1, and the line S_1T_1 being at right angles to S_1P_1. Then will

$$P_1Q_1 = ds,\ \angle S_1P_1T_1 = \phi,\ S_1Y_1 = p,\ S_1T_1 = v,\ \text{area } S_1P_1Q_1 = dA.$$

The truth of this proposition is manifest from the formulæ already established,

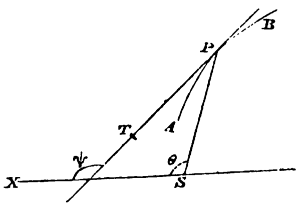
$$\sin\phi \,.\, ds = rd\theta, \quad \cos\phi \,.\, ds = dr,$$

$$p = r\sin\phi, \quad v = r\tan\phi, \quad dA = \tfrac{1}{2}r^2 d\theta.$$

We have only to remember this diagram in order to be able to call to mind all the polar formulæ of tangency.

Radius of Curvature in terms of r *and* p.

155. Let ψ denote the angle between the tangent TP of a curve AB, and the line SX.



Then
$$p = r \sin \phi,$$
$$dp = dr \sin \phi + r \cos \phi \, d\phi ;$$
or, multiplying by ds and observing that $\sin \phi \, ds = rd\theta$, and $\cos \phi \, ds = dr$,
$$dp \, ds = rd\theta dr + rd\phi dr = rdr \, (d\theta + d\phi) = rdrd\psi.$$

But we know that, ρ denoting the radius of curvature at P, $\rho = \frac{ds}{d\psi}$; hence
$$\rho = \frac{rdr}{dp} .$$

Chord of Curvature through the Pole.

156. Let the radius vector PS, produced if necessary, meet the osculating circle at P in a point p; and let $Pp = q$. Then it is plain that
$$q = 2\rho \sin \phi$$
$$= 2 \frac{rdr}{dp} . \sin \phi$$
$$= 2 \frac{pdr}{dp} .$$

Radius of Curvature in terms of r *and* θ.

157.* We have
$$\psi = \phi + \theta = \tan^{-1} \left(\frac{rd\theta}{dr} \right) + \theta,$$
and therefore, considering $d\theta$ constant,
$$d\psi = d\theta \left\{ \frac{d\left(\frac{r}{dr}\right)}{1 + \frac{r^2 d\theta^2}{dr^2}} + 1 \right\}$$
$$= d\theta \left(\frac{dr^2 - rd^2r}{r^2 d\theta^2 + dr^2} + 1 \right) = d\theta \frac{2dr^2 + r^2 d\theta^2 - rd^2r}{r^2 d\theta^2 + dr^2} .$$

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* This method of investigating the expression for ρ is given in the *Cambridge Mathematical Journal* for November, 1840.

Hence $$\rho = \frac{ds}{d\psi} = \frac{(r^2d\theta^2 + dr^2)^{\frac{1}{2}}}{d\psi} = \frac{(r^2d\theta^2 + dr^2)^{\frac{3}{2}}}{d\theta\,(2dr^2 + r^2d\theta^2 - rd^2r)}$$

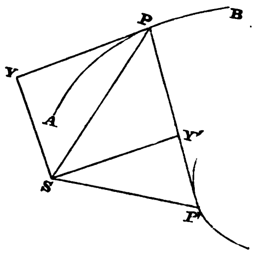
$$= \frac{\left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 + 2\frac{dr^2}{d\theta^2} - r\frac{d^2r}{d\theta^2}}.$$

The expression for ρ may be written, in a form perhaps rather less difficult to be remembered,

$$\rho = \frac{ds^3}{d\theta\,(dr^2 + ds^2 - rd^2r)}.$$

Evolutes of Polar Curves.

158. Let P' be the centre of curvature at any point P of a curve AB referred to polar coordinates. The line PP' will,



as we know, be a tangent to the locus of P'. From S draw SY' at right angles to PP' and join SP'. Let $SP' = r'$, $SY' = p'$.

Then, since $$PP' = \frac{rdr}{dp},$$

we see that $$r'^2 - p'^2 = \left(\frac{rdr}{dp} - p\right)^2 \ldots\ldots\ldots\ldots (1).$$

Also $$p^2 + p'^2 = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Between these two equations and the relation between p and r deducible from the equation to the involute, we may eliminate p and r, and thus obtain an equation between p' and r' which will determine the nature of the evolute. Conversely, having given the equation to the evolute, or an equation between p' and r', we may eliminate p' and r', and obtain an equation in p and r for the determination of the involute.

Ex. Let $p^2 = r^2 - a^2$: then

$$\frac{rdr}{dp} = p,$$

and therefore, from (1),

$$r'^2 = p'^2.$$

Also, from (2) and the proposed equation,

$$p'^2 = a^2.$$

Since

$$p'^2 = r'^2 = a^2,$$

it follows that the evolute is a circle of which the radius is a and of which the centre is at the pole.

Asymptotes.

159. An asymptote is a tangent to a curve, at a point infinitely distant, which passes within a finite distance from the origin of coordinates. Let then

$$f(\theta, \rho) = 0$$

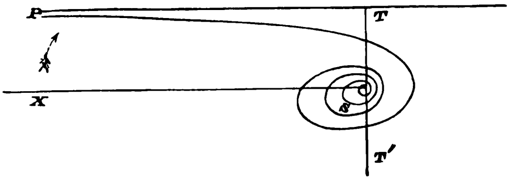
be the polar equation to a curve. Assume $\rho = \infty$, and obtain from this equation any corresponding value of θ, if there be any such. Then ascertain whether the corresponding value of the expression $r^2 \frac{d\theta}{dr}$ for the subtangent is finite, or infinite. If it be finite, a condition necessary for the existence of an asymptote, there will be an asymptote corresponding to the value assigned to θ. The asymptote will be constructed in the following manner. First draw the indefinite line SP inclined at the proper angle to the fixed line SX: from S draw ST at right angles to SP and equal to the value of $r^2 \frac{d\theta}{dr}$ for the particular value of θ: through T draw a line parallel to SP; this line will be an asymptote. If there be several values of θ, when $\rho = \infty$, which make $r^2 \frac{d\theta}{dr}$ finite, there will be several asymptotes.

Ex. 1. To find whether the curve $r = \dfrac{a}{\theta}$ has an asymptote.

Assume $r = \infty$: then $\theta = 0$. In this case

$$v = -\frac{r^2 d\theta}{dr} = a.$$

Hence there is an asymptote parallel to the line SX. Through S draw $ST = a$, and draw TP parallel to SX; TP will be the required asymptote. Had the value of v been negative instead of positive, we ought to have taken ST', at right angles to SX, equal to a, and drawn a line through T', parallel to SX, for the asymptote. The positive direction of θ is indicated by the arrow; we have considered only the positive values.



Ex. 2. To find whether the curve

$$r = \frac{a\theta^2}{\theta^2 - 1}$$

has asymptotes.

Assume $r = \infty$: then $\theta = \pm 1$.

Also $$\frac{1}{r} = \frac{1}{a}\left(1 - \frac{1}{\theta^2}\right),$$

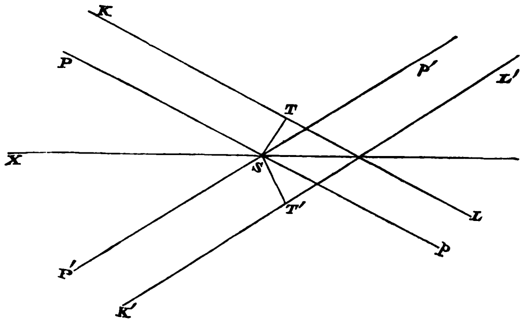
$$-\frac{dr}{r^2 d\theta} = \frac{2}{a\theta^3},$$

and therefore $$v = \tfrac{1}{2}a\theta^3.$$

If $\theta = 1$, $v = \frac{1}{2}a$; if $\theta = -1$, $v = -\frac{1}{2}a$. Hence the curve has two asymptotes.

Through S draw Pp, $P'p'$, inclined at angles, each of a circular measure unity, on opposite sides of SX. Draw $ST = \frac{1}{2}a$ at right angles to Pp, and $ST' = \frac{1}{2}a$ at right angles to $P'p'$. Through

T, T', draw KL, $K'L'$, parallel respectively to Pp, $P'p'$. Then



KL, $K'L'$, will be asymptotes.

Asymptotic Circles.

160. If, for any finite value a of ρ, θ becomes infinite, the curve will have an asymptotic circle, that is, a circle to which, as θ keeps increasing, the curve continually approaches without ever actually meeting.

Ex. 1. Take the curve

$$r = \frac{a\theta^2}{\theta^2 - 1}:$$

then, when $\theta = \infty$, $r = a$: when θ has any finite value greater than unity, r is greater than a. Hence the circle of which $r = a$ is the equation, is an interior asymptote.

Ex. 2. In the case of the curve

$$r = \frac{a\theta^2}{\theta^2 + 1},$$

$r = a$ is the equation to an exterior asymptotic circle.

Conditions for the Concavity and Convexity of the Curve towards the Pole and for Points of Inflection.

161. When the curve is concave or convex towards the pole in the neighbourhood of any point, it is easy to see that p increases or decreases respectively as r increases, the converse

proposition being likewise true. Hence, if $\frac{dp}{dr}$ be positive, the curve is concave towards the pole, and convex if it be negative.

The expression $\frac{dp}{dr}$ must therefore change sign as we pass through a point of inflection. To ascertain then a point of inflection we must assume

$$\frac{dp}{dr} = 0, \quad \text{or} = \infty,$$

and ascertain whether any values of r, obtained from either of these assumptions, corresponds to a point in passing through which we observe a change of sign in $\frac{dp}{dr}$.

CHAPTER XI.

ON THE METHODS OF TRACING THE FORMS OF CURVES FROM THEIR EQUATIONS.

General Principles and Examples.

162. From any proposed equation between x and y, the coordinates of the points which constitute a curvilinear locus, the form of the curve may be ascertained. The following is a general sketch of the ordinary method of effecting this object. First ascertain those values of x and y, if there be such, which render y and x respectively either zero or infinity: we shall thus determine where the curve cuts the axes of coordinates and the positions of asymptotes parallel to them. It will frequently be desirable next to ascertain the angles at which the curve cuts the axes. We must then determine the position of those asymptotes which are not parallel to either of the coordinate axes. The determination of the position of maximum or minimum ordinates and of points of inflection, as well as of multiple points and cusps, will give additional precision to the diagram. If our object be merely to ascertain the general form of the curve, it is not usually necessary to determine the actual position of points of inflection or maximum or minimum ordinates, the algebraical discussion of the equation being ordinarily sufficient to give a general notion of their position.

The method of tracing polar curves from their equations is similar, mutatis mutandis, to that which we have described in relation to curves referred to two axes of coordinates.

Ex. 1. To trace the curve represented by the equation

$$x^2y + aby - a^2x = 0.$$

Obtaining y explicitly in terms of x, we have

$$y = \frac{a^2x}{x^2 + ab}.$$

When $y = 0$, then $x = 0$ or $= \infty$. Hence the curve passes through the origin and touches the axis of x asymptotically.

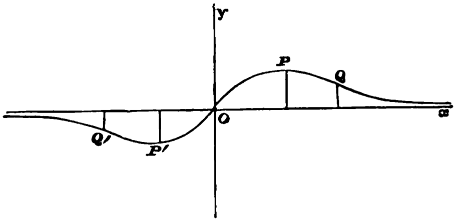
Let ϕ = the angle at which the curve cuts the axis of x at the origin: then, supposing x to be equal to zero,

$$\tan \phi = \frac{y}{x} = \frac{a^2}{x^2 + ab} = \frac{a}{b},$$

which shews that $\phi = \tan^{-1} \frac{a}{b}$, a positive angle depending upon the ratio between a and b.

Also, y is positive or negative accordingly as x is positive or negative. It is evident also that so long as x is finite y is finite.

The form of the curve will therefore be such as in the subjoined diagram.



Since y is zero when $x = -\infty$, $= 0$, and $= +\infty$, it is plain that y must have two geometrical maxima. Their exact position may be found by differentiation, or thus: from the equation to the curve there is

$$4x^2y^2 - 4a^2xy + a^4 = a^4 - 4aby^2,$$

$$(2xy - a^2)^2 = a^4 - 4aby^2.$$

In order that x may be possible, the left-hand side of this equation must not be negative; hence the greatest value of y, without reference to sign, is given by the equation

$$a^4 - 4aby^2 = 0.$$

Hence $$y = \frac{a^{\frac{3}{2}}}{2b^{\frac{1}{2}}}, \quad x = a^{\frac{1}{2}}b^{\frac{1}{2}},$$

and $$y = -\frac{a^{\frac{3}{2}}}{2b^{\frac{1}{2}}}, \quad x = -a^{\frac{1}{2}}b^{\frac{1}{2}},$$

determine two points P, P' of the curve, the ordinates of which are geometrical maxima. The value $-\dfrac{a^{\frac{3}{2}}}{2b^{\frac{1}{2}}}$ is an analytical minimum of y, being its greatest negative value. It is easily seen from the general form of the curve that there must be a point of inflection at the origin of coordinates, and two others between $x = -\infty$, $x = 0$, and $x = 0$, $x = +\infty$. We will however prove this analytically, and determine the actual position of these points.

Differentiating twice, we get
$$\frac{dy}{dx} = a^2 \frac{ab - x^2}{(x^2 + ab)^2},$$
$$\frac{d^2y}{dx^2} = 2a^2 . \frac{x(x^2 - 3ab)}{(x^2 + ab)^3};$$

from this result it appears that there are three points of inflection, one at the origin, and two others Q, Q', of which the coordinates are
$$x = (3ab)^{\frac{1}{2}}, \quad y = \frac{3^{\frac{1}{2}}a^{\frac{3}{2}}}{4b^{\frac{1}{2}}},$$

and $$x = -(3ab)^{\frac{1}{2}}, \quad y = -\frac{3^{\frac{1}{2}}a^{\frac{3}{2}}}{4b^{\frac{1}{2}}}.$$

This curve was called the Anguinea by Newton, in consequence of its form, and is one of the seventy-two species of curves of the third order which he has enumerated.

Ex. 2. To trace the curve
$$x^3 - axy - b^2y = 0.$$

Making y explicit, we have

$$y = \frac{x^3}{ax + b^2}.$$

When $x = -\frac{b^2}{a}$, $y = \infty$: hence

$$ax + b^2 = 0$$

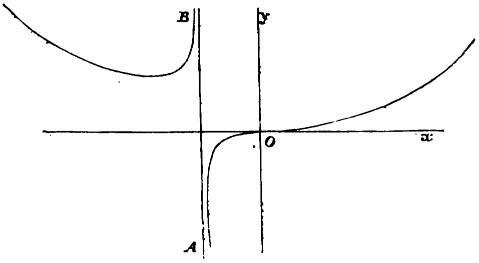
is the equation to an asymptote, AB.

When $x = 0$, $y = 0$, and

$$\frac{y}{x} = \frac{x^2}{ax + b^2} = 0,$$

which shews that the curve touches the axis of x at the origin.

When x has any positive value or any negative value greater than $\frac{b^2}{a}$, y is positive, being negative from $x = -\frac{b^2}{a}$ to $x = 0$. When $x = \pm\infty$, $y = +\infty$. This curve is one of a class to which the name of Trident has been given by Newton: its form is indicated in the following diagram.



Ex. 3. To trace the curve

$$y^2 = \frac{(x + a)^2 (b^2 - x^2)}{x^2},$$

a being supposed to be less than b.

When $x = +0$ or $= -0$, $y = \pm\infty$, which shews that the axis of y is an asymptote. When $x = -a$, $y = 0$; when $x = \pm b$, $y = 0$. If x have a greater value than b, without reference to sign, y will be impossible. For each value of x, there will be two equal values of y with opposite signs, which shews that the curve is symmetrical with relation to the axis of x.

Let α, β, be the angles at which the curve cuts the axis of x, viz. when $x = -a$, or $= \pm b$ respectively: then

$$\tan^2 \alpha = \text{limit of } \frac{y^2}{(x+a)^2}, \quad \text{when } x = -a,$$

$$= \text{limit of } \frac{b^2 - x^2}{x^2} = \frac{b^2 - a^2}{a^2},$$

which determines two equal values for α with opposite signs.

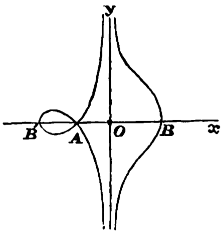
Also $$\tan^2 \beta = \text{limit of } \frac{y^2}{(b \mp x)^2}, \text{ when } x = \pm b,$$

$$= \text{limit of } \frac{(x+a)^2 (b \pm x)}{x^2 (b \mp x)}$$

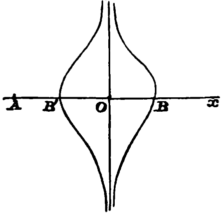
$$= \infty,$$

which shews that $\beta = \frac{\pi}{2}$.

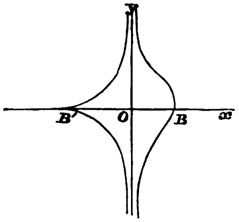
This curve is the Conchoid of Nicomedes: its general form is indicated in the following diagram, where $OA = a$, $OB = OB' = b$.



If a had been supposed to be greater than b, the curve would have had the following form, the point A being a conjugate point.



If $a = b$, the curve will have the following form, there being a cusp at B'.



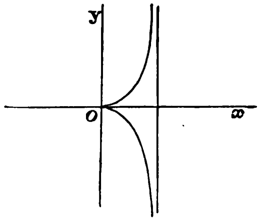
Ex. 4. To trace the Cissoid of Diocles from its equation

$$y^2 = \frac{x^3}{a - x}.$$

If x be negative or greater than a, y is impossible. If $x = a$, then $y = \pm \infty$, and therefore $x = a$ is the equation to an asymptote. Since for each value of x there are two equal values of y with opposite signs, it follows that the curve must be symmetrical in regard to the axis of x. Also, when $x = 0$,

$$\frac{y^2}{x^2} = \frac{x}{a - x} = 0,$$

and therefore there is a cusp at the origin. The form of the curve is the following.



Ex. 5. To trace the Lemniscata of James Bernoulli from its equation

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2).$$

This may be effected most readily by transforming the equation to polar coordinates. Thus, putting $x = r \cos \theta$, $y = r \sin \theta$, the equation becomes

$$r^2 = a^2 (\cos^2 \theta - \sin^2 \theta) = a^2 \cos 2\theta.$$

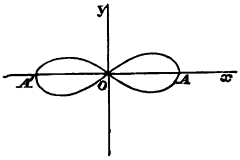
When $\theta = 0$, $r = \pm a$, which shews that the curve cuts the axis of x at two points A, A', equidistant from O. When $\theta = \frac{\pi}{4}$, $r = 0$, and when $\theta > \frac{\pi}{4}$ and $< \frac{3\pi}{4}$, r is impossible. For each value of θ, r has two equal values with opposite signs, and r has the same values when $-\theta$ is substituted for $+\theta$. The curve is therefore symmetrical in regard to both axes of coordinates.

Also $$2 \log r = 2 \log a + \log \cos 2\theta,$$

$$2 \frac{dr}{r\,d\theta} = \frac{-2 \sin 2\theta}{\cos 2\theta},$$

$$r \frac{d\theta}{dr} = -\cot 2\theta = \infty, \quad \text{when } \theta = 0.$$

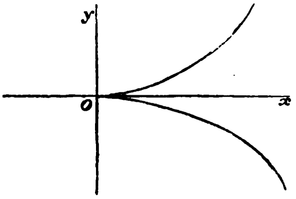
Hence the curve cuts the axis of x at right angles at A and A'. The form of the curve is exhibited in the diagram



Ex. 6. To trace the semi-cubical parabola, of which the equation is

$$ay^2 = x^3.$$

Its form is the following.

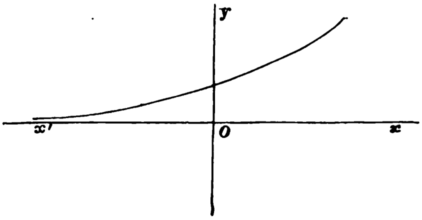


Ex. 7. To trace the logarithmic curve of James Gregorie, of which the equation is

$$y = a^x.$$

The form of the curve is expressed in the diagram, Ox or

Ox' being the positive direction of the axis of x accordingly as a is greater or less than unity.



Ex. 8. To trace the Quadratrix of Tschirnhausen from its equation

$$y = \sin \frac{\pi x}{2}.$$

When $x = 2\lambda + 1$, λ being any integer,

$$y = \sin \left\{(2\lambda + 1)\frac{\pi}{2}\right\} = \cos \pi\lambda = (-1)^\lambda,$$

which shews that y is equal to $+1$ when x has any one of the series of values 1, 5, 9, ..., and to -1, when x has any one of the series of values 3, 7, 11, ...

Also, when $x = 2\lambda$, λ being any integer,

$$y = \sin \lambda\pi = 0,$$

or the curve cuts the axis of x at distances

$$\ldots\ldots -6,\ -4,\ -2,\ 0,\ 2,\ 4,\ 6, \ldots\ldots$$

from the origin of coordinates.

Also, when $y = 0$, or $x = 2\lambda$,

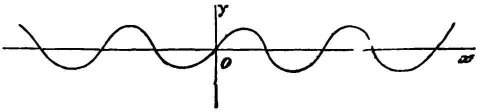
$$\frac{dy}{dx} = \frac{\pi}{2} \cos \frac{\pi x}{2} = \frac{\pi}{2}(-1)^\lambda,$$

and therefore the angles at which the curve cuts the axis of x are equal and alternately positive and negative. We may remark also that, when $x = 2\lambda + 1$,

$$\frac{dy}{dx} = \frac{\pi}{2} \cos \left(\frac{\pi}{2} + \lambda\pi\right) = 0,$$

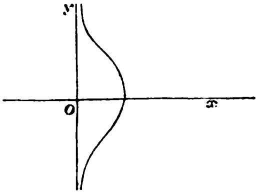
or the tangent is parallel to the axis of x.

The form of the curve, from the preceding conclusions, is evidently the following, being infinite both in the positive and in the negative direction of the axis of x.



Ex. 9. To trace the Versiera or Witch of Donna Maria Agnesi from its equation

$$xy^2 = a^2(a - x).$$

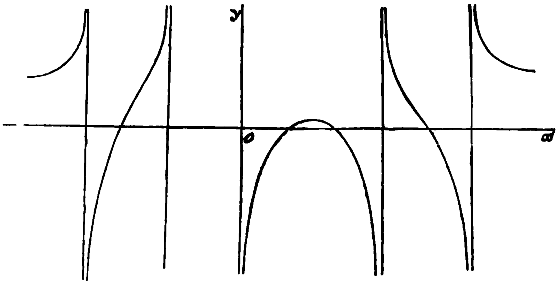


There are two points of inflection corresponding to the value $\frac{3}{2}a$ of x.

Ex. 10. To trace the Quadratrix of Dinostratus from its equation

$$y = (a - x) \tan \frac{\pi x}{2a}.$$

This curve cuts the axis of x at a series of points, the distances of which from the origin are represented by the expression $2\lambda a$, where λ is any integer, positive or negative.



It has also a series of asymptotes parallel to the axis of y, and at the general distance $(2\lambda + 1)a$ from the origin, λ being any integer positive or negative, but not zero.

Ex. 11. To trace the Cardioid from its polar equation

$$r = a(1 - \cos\theta).$$

It is evident that, as θ increases from 0 to π, r increases from 0 to $2a$; and that the expression for r remains unaltered when $-\theta$ is substituted for $+\theta$. Also

$$\log r = \log a + \log(1 - \cos\theta),$$

$$\frac{dr}{r\,d\theta} = \frac{\sin\theta}{1 - \cos\theta},$$

and therefore

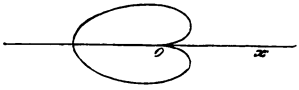
$$r\frac{d\theta}{dr} = \frac{1 - \cos\theta}{\sin\theta}$$

$$= \tan\frac{\theta}{2}$$

$$= 0, \quad \text{when } \theta = 0,$$

$$= \infty, \quad \text{when } \theta = \pi.$$

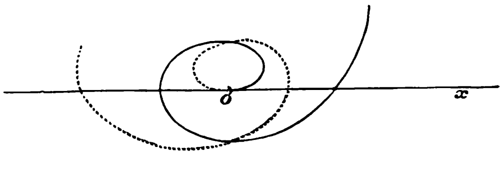
Hence the form of the curve must be such as that exhibited in the diagram, there being a ceratoid cusp at the origin of polar coordinates.



Ex. 12. To trace the spiral of Conon or Archimedes from its equation

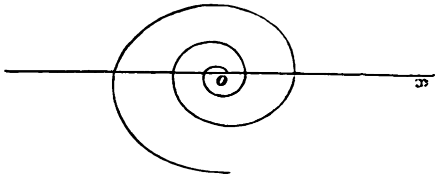
$$r = a\theta.$$

The dotted line in the figure indicates that portion of the curve which is due to the negative values of θ.



Ex. 13. To trace the Logarithmic Spiral of Descartes from its equation

$$r = c\,\epsilon^{\frac{\theta}{a}}.$$



This curve is called also the Equiangular Spiral, because it cuts all its radii at the same angle: in fact

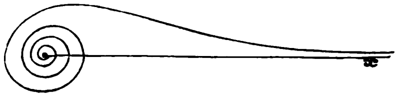
$$\log r = \log c + \frac{\theta}{a},$$

$$\frac{dr}{r\,d\theta} = \frac{1}{a}, \quad \frac{r\,d\theta}{dr} = a, \text{ a constant quantity.}$$

Ex. 14. To trace the Lituus from its equation

$$r^2 = \frac{a^2}{\theta}.$$

This curve touches Ox asymptotically, and approaches O by an infinite number of circumvolutions.



For methods of constructing geometrically the curves which have been above considered, and deducing their equations from their geometrical properties, which is the converse of the course which we have adopted, as well as for historical information respecting them, the student is referred to Peacock's or Gregory's *Examples*.

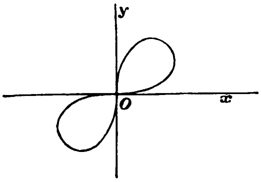
Ex. 15. To trace the curve represented by the equation

$$x^4 + y^4 = a^2xy.$$

It is evident that x and y must have the same sign: hence the curve can lie only in two quadrants. In the neighbourhood of the origin, neglecting small quantities of higher orders than the second, we have

$$xy = 0,$$

which shews that the axes of x and y are both touched by a branch passing through the origin. If $-x$ and $-y$ be written for x and y, the equation is not altered, and therefore the curve is the same in both quadrants. It is impossible for either x or y to be infinite, since $x^4 + y^4$ would then be a positive quantity of an infinitely higher order of magnitude than a^2xy. The curve must therefore be of the form



Ex. 16. To trace the curve represented by the equation

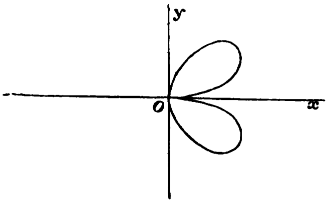
$$x^4 + y^4 = 2axy^2.$$

At the origin there is

$$xy^2 = 0,$$

which shews that the axis of x is touched by two branches and the axis of y by one branch of the curve. It is evidently impossible for x to have a negative value. The curve is symmetrical with respect to the axis of x, because its equation remains the same when $-y$ is substituted for $+y$. Neither x nor y can be infinite, since $x^4 + y^4$ would then be infinite compared with $2axy^2$.

Hence the curve must be of the form



Homogeneous Curves.

163. Curves represented by equations of the form

$$u = c,$$

where u is a homogeneous function of x and y, and where c is a constant quantity, may be traced very conveniently by assuming $y = tx$, and obtaining x and y in terms of t: a series of values must be assigned to t, and the corresponding values of x and y must be tabulated. It is desirable, however, first to ascertain whether there be any asymptotes by the method of Art. (110), Chap. II.

Ex. 1. To trace the curve

$$x^5y - xy^5 = a^6.$$

Putting $x = \frac{\lambda}{r}$, $y = \frac{\mu}{r}$, we get

$$\lambda^5\mu - \lambda\mu^5 = a^6r^6 \ldots\ldots\ldots\ldots\ldots (1),$$

and $$\mu(5\lambda^4 - \mu^4)\frac{d\lambda}{dr} - \lambda(5\mu^4 - \lambda^4)\frac{d\mu}{dr} = 6a^6r^5 \ldots\ldots (2).$$

From (1) we see that

$$(\lambda)(\mu)\{(\lambda)^4 - (\mu)^4\} = 0,$$

and, from (2),

$$(\mu)\{5(\lambda)^4 - (\mu)^4\}\left(\frac{d\lambda}{dr}\right) - (\lambda)\{5(\mu)^4 - (\lambda)^4\}\left(\frac{d\mu}{dr}\right) = 0.$$

Hence we have the four systems

$$\left\{\begin{matrix}(\lambda) = 0 \\ \left(\frac{d\lambda}{dr}\right) = 0\end{matrix}\right\}, \quad \left\{\begin{matrix}(\mu) = 0 \\ \left(\frac{d\mu}{dr}\right) = 0\end{matrix}\right\}, \quad \left\{\begin{matrix}(\lambda) = (\mu) \\ \left(\frac{d\lambda}{dr}\right) - \left(\frac{d\mu}{dr}\right) = 0\end{matrix}\right\},$$

$$\left\{\begin{matrix}(\lambda) + (\mu) = 0 \\ \left(\frac{d\lambda}{dr}\right) + \left(\frac{d\mu}{dr}\right) = 0\end{matrix}\right\}.$$

Hence, from the equation

$$(\lambda)y' - (\mu)x' = (\lambda)\left(\frac{d\mu}{dr}\right) - (\mu)\left(\frac{d\lambda}{dr}\right),$$

we obtain four asymptotes to the curve, represented by the equations

$$x' = 0, \quad y' = 0, \quad y' - x' = 0, \quad y' + x' = 0.$$

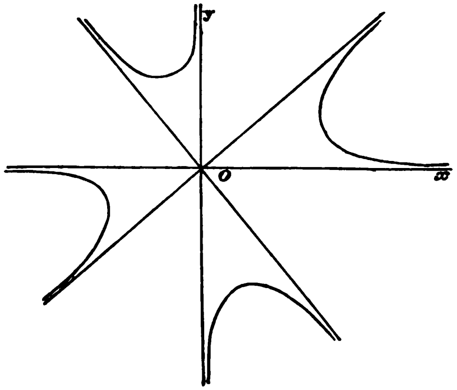
Again, putting $y = tx$, we see that

$$x^6 = \frac{a^6}{t(1-t^4)}, \quad y^6 = \frac{a^6 t^5}{1-t^4},$$

and therefore, observing that the ratio of y to x must be always of the same sign as t, we have the following table of corresponding values:

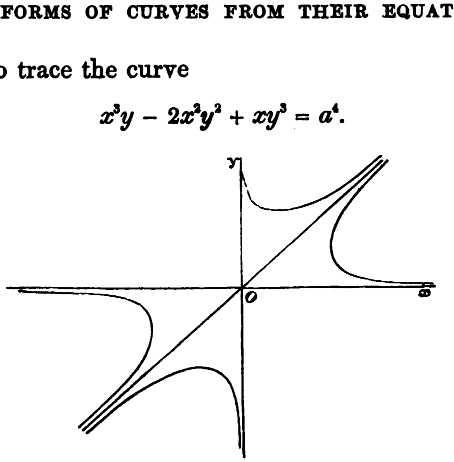
t	x	y
$+0$	$+\infty$	$+0$
$+0$	$-\infty$	-0
$1-0$	$+\infty$	$+\infty$
$1-0$	$-\infty$	$-\infty$
$-\infty$	$+0$	$-\infty$
$-\infty$	-0	$+\infty$
$-(1+0)$	$+\infty$	$-\infty$
$-(1+0)$	$-\infty$	$+\infty$

The form of the curve will therefore be the following.



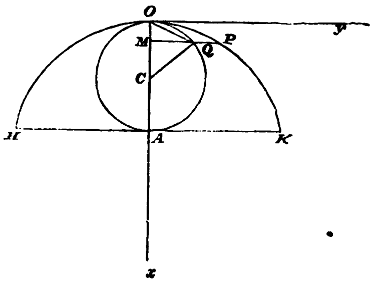
Ex. 2. To trace the curve

$$x^3y - 2x^2y^2 + xy^3 = a^4.$$



The Cycloid.

164. As an example of deducing the equation to a curve from its geometrical definition, which is exactly the converse of tracing a curve from its equation, we will investigate the equation to the cycloid from the nature of its generation.



Let C be the centre of a circle in contact at A with the straight line HK. Let O be the extremity of the diameter through A. Suppose this circle to roll, without sliding, along HK; the point O of the circumference will then trace out a curve OPK, which is called the Cycloid. Let OAx be taken as the axis of x, and Oy, at right angles to OA, as the axis of y. Suppose that, when O has arrived at a point P of the cycloid, the circle has revolved about its centre through an angle θ; then its centre must have advanced, parallel to

HK, through a space $a\theta$, a being the radius of the circle: for, since the circle rolls without sliding, it follows that the velocity of its point of contact, parallel to KH, due to its rotation about C, must be equal to the velocity of its point of contact, parallel to HK, due to the translation of C.

From C draw CQ, making $\angle OCQ = \theta$: draw MQ at right angles to AO, and produce MQ to P, making QP equal to $a\theta$. Then Q will be the position into which O would be carried by the rotation alone, QP being its additional progress due to the translation of C. Let $OM = x$, $PM = y$. Then

$$x = OC - CM = a - a\cos\theta = a(1 - \cos\theta),$$

and $$y = PQ + QM = a\theta + a\sin\theta = a(\theta + \sin\theta):$$

eliminating θ, we shall get

$$y = a \operatorname{vers}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

which is the equation to the cycloid.

The curve will evidently be symmetrical on both sides of the axis of x: for if we put $-\theta$ for θ, we see that y retains the same magnitude with an opposite sign, and x remains entirely unchanged.

If $\theta = \pi$, $y = AK = \pi a = AH$. Also arc $OQ = a\theta = PQ$.

If for θ we write $2\lambda\pi + \theta$, λ being any integer whatever, the expression $a(1 - \cos\theta)$ remains unchanged, while $a(\theta + \sin\theta)$ receives an increment $2\lambda\pi a$. This shews that the two equations between θ, x, y, represent a series of similar, equal, and similarly situated cycloids, with their vertices arranged along the axis of y, both in the positive and negative directions, at intervals of $2\pi a$.

Tangent and Normal to the Cycloid.

165. The general equation to the tangent of a curve is

$$x'dy - y'dx = xdy - ydx:$$

this equation becomes, for the cycloid,

$$x'(1 + \cos\theta) - y'\sin\theta = (1 - \cos\theta)(1 + \cos\theta) - (\theta + \sin\theta)\sin\theta$$
$$= -\theta\sin\theta.$$

If ϕ be the inclination of the tangent to the axis of x, then

$$\tan\phi = \frac{dy}{dx} = \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta},$$

a result which shews that the tangent at P is parallel to the chord OQ, and that consequently the normal at P is parallel to the radius CQ.

Arc of the Cycloid.

166. Differentiating the formulæ for x and y, we get

$$\begin{aligned} ds^2 = dx^2 + dy^2 &= a^2\sin^2\theta\,d\theta^2 + a^2(1+\cos\theta)^2\,d\theta^2 \\ &= 2a^2d\theta^2\,(1+\cos\theta) \\ &= \frac{2a^2\sin^2\theta\,d\theta^2}{1-\cos\theta} \\ &= \frac{2a\,dx^2}{x}\,; \end{aligned}$$

$$ds = \left(\frac{2a}{x}\right)^{\frac{1}{2}} dx, \qquad s = (8ax)^{\frac{1}{2}},$$

or
$$s^2 = 8ax.$$

Let c = the chord of the arc OQ; then, by the nature of the circle,

$$c^2 = 2ax:$$

hence
$$s^2 = 4c^2,$$

or
$$\text{arc } OP = 2 \text{ chord } OQ.$$

Radius of Curvature of the Cycloid.

167. By Art. (132), we have

$$\rho^2 = \frac{(dx^2+dy^2)^3}{(dx\,d^2y - dy\,d^2x)^2}.$$

Taking θ as the independent variable,

$$dx = a\sin\theta\,d\theta, \qquad d^2x = a\cos\theta\,d\theta^2,$$

$$dy = a(1+\cos\theta)\,d\theta, \qquad d^2y = -a\sin\theta\,d\theta^2:$$

hence $$dx^2 + dy^2 = 2a^2 d\theta^2 (1 + \cos\theta),$$

$$dxd^2y - dyd^2x = - a^2 d\theta^3 (1 + \cos\theta);$$

and therefore

$$\rho^2 = 8a^2 (1 + \cos\theta) = 8a(2a - x).$$

Evolute of the Cycloid.

168. If α, β, be the coordinates of any point of the evolute, then, by Art. (136),

$$\alpha dx + \beta dy = xdx + ydy,$$

and $$\alpha d^2x + \beta d^2y = xd^2x + yd^2y + dx^2 + dy^2.$$

From these two equations there is

$$\alpha (dxd^2y - dyd^2x) = x (dxd^2y - dyd^2x) - dy (dx^2 + dy^2) \ldots (1),$$

and $$\beta (dyd^2x - dxd^2y) = y (dyd^2x - dxd^2y) - dx (dy^2 + dx^2) \ldots (2).$$

From (1) we have, expressing x and y in terms of θ,

$$-\alpha (1 + \cos\theta) = - a (1 - \cos\theta)(1 + \cos\theta) - a (1 + \cos\theta) . 2 (1 + \cos\theta),$$

hence $$\alpha = a (1 - \cos\theta) + 2a (1 + \cos\theta) = a (3 + \cos\theta),$$

or $$\alpha - 2a = a (1 + \cos\theta) \ldots\ldots\ldots\ldots (3).$$

From (2), we have

$$\beta (1 + \cos\theta) = a (\theta + \sin\theta)(1 + \cos\theta) - \sin\theta . 2a (1 + \cos\theta),$$

or $$\beta = a (\theta + \sin\theta) - 2 \sin\theta = a (\theta - \sin\theta) \ldots . (4).$$

Putting $\theta = \phi \pm \pi$ in (3) and (4), we have

$$\alpha - 2a = a (1 - \cos\phi) \ldots\ldots\ldots\ldots (5),$$

and $$\beta = a (\phi \pm \pi + \sin\phi),$$

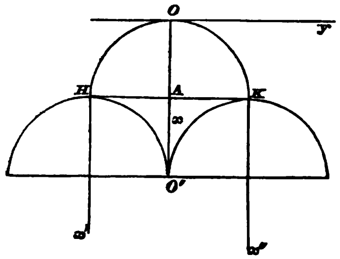
or $$\beta \mp \pi a = a (\phi + \sin\phi) \ldots\ldots\ldots\ldots (6).$$

If we now change the origin to a point $2a$, $\pm \pi a$, by putting $\alpha = \alpha' + 2a$, $\beta = \beta' \pm \pi a$, in (5), (6), we get

$$\alpha' = a (1 - \cos\phi) \ldots\ldots\ldots\ldots (7),$$

and $$\beta' = a (\phi + \sin\phi) \ldots\ldots\ldots\ldots (8).$$

These results shew that the evolute of the cycloid *HOK* is a portion of the locus consisting of the infinite series of cycloids denoted by the equations (7) and (8). Two of this series have their vertices at *H* and *K*, and their point of junction *O′*, where they form a cusp, in the line *OA* produced, *AO′* being equal to *AO*. It is evident that each of the cycloidal evolutes is similar, and equal to the original cycloid, and similarly situated. *HO′K* is evidently the portion of the locus of (7) and (8) which constitutes the evolute of *HOK*.



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